Solution of Different Types of Differential Equations

4 Marks Questions

1. Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$, given that $y = 0$ when $x = 1$. **All India 2014**

Given differential equation is

$$\frac{dy}{dx} = 1 + x + y + xy$$

$$\Rightarrow \quad \frac{dy}{dx} = 1(1 + x) + y(1 + x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + x)(1 + y) \quad \text{...(i)} \quad \textbf{(1)}$$

On separating variables, we get

$$\frac{1}{(1 + y)} dy = (1 + x)\, dx \quad \text{...(ii)}$$

On integrating both sides of Eq. (ii), we get

$$\int \frac{1}{1 + y} dy = \int (1 + x)\, dx$$

$\Rightarrow \quad \log|1+y| = x + \frac{x^2}{2} + C \quad$ ...(iii) **(1)**

Also, given that $y = 0$, when $x = 1$.

On substituting $x = 1, y = 0$ in Eq. (iii), we get

$\log|1+0| = 1 + \frac{1}{2} + C \Rightarrow C = -\frac{3}{2}$ $[\because \log 1 = 0]$

(1)

Now, on substituting the value of C in Eq. (iii), we get

$$\log|1+y| = x + \frac{x^2}{2} - \frac{3}{2}$$

which is the required particular solution of given differential equation. **(1)**

2. Find the particular solution of the differential equation $x\frac{dy}{dx} - y + x \operatorname{cosec}\left(\frac{y}{x}\right) = 0$ or $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$, given that $y = 0$, when $x = 1$. **All India 2014C, 2011; Delhi 2009**

Given differential equation is

$$x\frac{dy}{dx} - y + x\,\text{cosec}\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} + \text{cosec}\left(\frac{y}{x}\right) = 0$$

Above equation can be written as

$$\frac{dy}{dx} = \frac{y}{x} - \text{cosec}\left(\frac{y}{x}\right) \quad \text{...(i)}$$

which is a homogeneous differential equation.

On putting $y = vx$,

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \text{ in Eq. (i), we get}$$

$$v + x\frac{dv}{dx} = \frac{vx}{x} - \text{cosec}\left(\frac{vx}{x}\right)$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \text{cosec}\, v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\text{cosec}\, v \Rightarrow \frac{dv}{\text{cosec}\, v} = \frac{-dx}{x} \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{dv}{\text{cosec}\, v} = \int -\frac{dx}{x}$$

$$\Rightarrow \int \sin v\, dv = \int -\frac{dx}{x} \qquad \left[\because \frac{1}{\text{cosec}\, v} = \sin v\right]$$

$$\Rightarrow \quad -\cos v = -\log|x| + C$$

$$\left[\because \int \sin x\, dx = -\cos x + C \text{ and } \int \frac{1}{x}\, dx = \log|x| + C\right]$$

On putting $v = \frac{y}{x}$, we get

$$-\cos\frac{y}{x} = -\log|x| + C$$

$$\Rightarrow \quad \cos\frac{y}{x} = +(\log|x| - C)$$

$$\Rightarrow \quad \frac{y}{x} = \cos^{-1}(\log|x| - C)$$

$$\Rightarrow \quad y = x\cos^{-1}(\log|x| - C) \quad \text{...(ii)}$$ **(1½)**

Also, given that $x = 1$ and $y = 0$.

On putting above values in Eq. (ii), we get

$$0 = 1\cos^{-1}(\log|1| - C)$$

$$\Rightarrow \quad \cos 0° = 0 - C$$

$$\Rightarrow \quad 1 = 0 - C$$

$$\Rightarrow \quad C = -1$$

$$\therefore \quad y = x\cos^{-1}(\log|x| + 1)$$ **(1½)**

which is required solution.

3. Solve the differential equation

$x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$. **Foreign 2014; Delhi 2009**

Given differential equation is

$$(x \log x) \cdot \frac{dy}{dx} + y = \frac{2}{x} \log x$$

On dividing both sides by $x \log x$, we get

$$\frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x^2} \frac{\log x}{\log x} = \frac{2}{x^2} \quad \text{...(i)}$$

which is a linear differential equation of first order and is of the form

$$\frac{dy}{dx} + Py = Q \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{1}{x \log x} \text{ and } Q = \frac{2}{x^2} \quad \textbf{(1)}$$

$$\therefore \quad IF = e^{\int \frac{1}{x \log x} dx} = e^{\log \log x}$$

$$\left[\text{for} \int \frac{1}{x \log x} dx \Rightarrow \text{put } \log x = t \Rightarrow \frac{1}{x} dx = dt \quad \therefore \int \frac{1}{t} dt = \log |t| = \log |\log x| \right]$$

$$\Rightarrow \quad IF = \log x \quad [\because e^{\log x} = x]$$

(1)

Now, solution of above equation is given by

$$y \times IF = \int (Q \times IF)\, dx + C \quad \text{...(iii)}$$

On putting IF $= \log x$ and $Q = \frac{2}{x^2}$ in Eq. (iii), we get

$$y \log x = \int \underset{\text{II}}{\frac{2}{x^2}} \underset{\text{I}}{\log x}\, dx$$

$$\Rightarrow \quad y \log x = \log x \int \frac{2}{x^2} dx - \int \left(\frac{d}{dx} (\log x) \cdot \int \frac{2}{x^2} dx \right) dx$$

[using integration by parts]

$$\Rightarrow \quad y \log x = \log x \cdot 2\left(-\frac{1}{x}\right) - \int \frac{1}{x} \cdot 2\left(-\frac{1}{x}\right) dx \quad \textbf{(1)}$$

$$\Rightarrow \quad y \log x = -\frac{2}{x} \log x - \int \frac{2}{x}\left(-\frac{1}{x}\right) dx$$

$$\Rightarrow \quad y \log x = -\frac{2}{x} \log x + \int \frac{2}{x^2} dx$$

$$\therefore \quad y \log x = -\frac{2}{x} \log x - \frac{2}{x} + C \quad \textbf{(1)}$$

which is the required solution.

4. Find the general solution of the differential equation $(x - y) \frac{dy}{dx} = x + 2y$.

Delhi 2014C; All India 2010

Given differential equation is

$$(x-y)\frac{dy}{dx} = x + 2y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x+2y}{x-y} \qquad \text{...(i)} \quad \textbf{(1)}$$

which is a homogeneous equation.

On putting $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{...(ii)}$$

in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{x+2vx}{x-vx} = \frac{1+2v}{1-v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+2v}{1-v} - v = \frac{1+2v-v+v^2}{1-v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v+v^2}{1-v}$$

$$\Rightarrow \quad \frac{1-v}{v^2+v+1}dv = \frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{1-v}{v^2+v+1}dv = \int \frac{dx}{x} \qquad \textbf{(1)}$$

$$\Rightarrow \quad I = \log|x| + C \qquad \text{...(iii)}$$

where, $\quad I = \int \frac{1-v}{v^2+v+1}dv$

Let $\quad 1 - v = A \cdot \frac{d}{dv}(v^2+v+1) + B$

$$\Rightarrow \quad 1 - v = A(2v+1) + B$$

On comparing coefficients of v and constant term from both sides, we get

$$2A = -1 \quad \Rightarrow \quad A = -\frac{1}{2} \quad \text{and} \quad A + B = 1$$

$$\Rightarrow -\frac{1}{2} + B = 1 \quad \Rightarrow \quad B = 1 + \frac{1}{2} \quad \Rightarrow \quad B = \frac{3}{2}$$

So, we write $1 - v = -\frac{1}{2}(2v+1) + \frac{3}{2}$

1 3

Then, $I = \int \frac{-\frac{1}{2}(2v+1) + \frac{3}{2}}{v^2 + v + 1} dv$

$\Rightarrow \quad I = -\frac{1}{2} \int \frac{2v+1}{v^2+v+1} dv + \frac{3}{2} \int \frac{dv}{v^2+v+1}$

$\Rightarrow \quad I = -\frac{1}{2} \log |v^2 + v + 1|$

$$+ \frac{3}{2} \int \frac{dv}{v^2 + v + 1 + \frac{1}{4} - \frac{1}{4}}$$

$$\left[\begin{aligned} &\because \int \frac{2v+1}{v^2+v+1} dv \Rightarrow \text{put } v^2 + v + 1 = t \\ &\qquad\qquad (2v+1)\, dv = dt \\ &\therefore \int \frac{dt}{t} = \log |t| + c = \log |v^2 + v + 1| + c \end{aligned} \right]$$

$$\Rightarrow I = -\frac{1}{2} \log |v^2 + v + 1| + \frac{3}{2} \int \frac{dv}{\left(v + \frac{1}{2}\right)^2 + \frac{3}{4}} \qquad \textbf{(1)}$$

$$\Rightarrow I = -\frac{1}{2} \log |v^2 + v + 1| + \frac{3}{2} \int \frac{dv}{\left(v + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$\Rightarrow I = -\frac{1}{2} \log |v^2 + v + 1|$

$$+ \frac{3}{2} \times \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{v + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + C$$

$$\left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \right]$$

$$\Rightarrow I = -\frac{1}{2}\log|v^2 + v + 1| + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) + C$$

On putting $v = \frac{y}{x}$, we get

$$I = -\frac{1}{2}\log\left|\frac{y^2}{x^2} + \frac{y}{x} + 1\right| + \sqrt{3}\tan^{-1}\left(\frac{\frac{2y}{x}+1}{\sqrt{3}}\right) + C$$

$$\left[\because y = vx \therefore v = \frac{y}{x}\right]$$

$$\Rightarrow I = -\frac{1}{2}\log\left|\frac{y^2 + xy + x^2}{x^2}\right| + \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + C$$

On putting the value of I in Eq. (iii), we get

$$-\frac{1}{2}\log\left|\frac{y^2 + xy + x^2}{x^2}\right| + \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)$$
$$= \log|x| + C$$

which is the required solution. **(1)**

5. Find the particular solution of the differential equation $\left\{x\sin^2\left(\frac{y}{x}\right) - y\right\}dx + x\,dy = 0$, given that $y = \frac{\pi}{4}$, when $x = 1$. **All India 2014C**

Given differential equation is

$$\left[x \sin^2\left(\frac{y}{x}\right) - y\right] dx + x\,dy = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y - x \sin^2\left(\frac{y}{x}\right)}{x} \qquad \text{...(i)}$$

which is a homogeneous differential equation.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + \frac{x\,dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{vx - x \sin^2\left(\frac{vx}{x}\right)}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \sin^2 v \Rightarrow x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \quad \operatorname{cosec}^2 v\,dv = -\frac{dx}{x} \qquad \textbf{(1)}$$

On integrating both sides, we get

$$\int \operatorname{cosec}^2 v\,dv + \int \frac{dx}{x} = 0$$

$$\Rightarrow \quad -\cot v + \log|x| = C$$

$$\Rightarrow -\cot\left(\frac{y}{x}\right) + \log|x| = C \qquad \left[\because v = \frac{y}{x}\right] \qquad \text{...(ii)}$$

(1)

Also, given that $y = \frac{\pi}{4}$, when $x = 1$.

On putting $x = 1$ and $y = \frac{\pi}{4}$ in Eq. (ii), we get

$$-\cot\left(\frac{\pi}{4}\right) + \log 1 = C$$

$$\Rightarrow \quad C = -1 \qquad \left[\because \cot\frac{\pi}{4} = 1\right] \textbf{(1)}$$

On putting this value of C in Eq. (ii), we get

$$-\cot\left(\frac{y}{x}\right) + \log|x| = 1$$

$$\Rightarrow \quad 1 + \log|x| - \cot\left(\frac{y}{x}\right) = 0$$

which is the required particular solution of given differential equation. **(1)**

6. Find the particular solution of the differential equation

$$\frac{dy}{dx} = \frac{x\,(2\log x + 1)}{\sin y + y\cos y},$$ given that $y = \frac{\pi}{2}$, when $x = 1$.

Delhi 2014

Given differential equation is

$$\frac{dy}{dx} = \frac{x(2 \log x + 1)}{\sin y + y \cos y}$$

On separating the variables, we get

$$(\sin y + y \cos y)\, dy = x(2 \log x + 1)\, dx$$

$$\Rightarrow \sin y\, dy + y \cos y\, dy = 2x \log x\, dx + x\, dx \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \sin y\, dy + \int \underset{\text{I}}{y} \underset{\text{II}}{\cos y}\, dy = 2 \int \underset{\text{II}}{x} \underset{\text{I}}{\log x}\, dx + \int x\, dx$$

$$\Rightarrow -\cos y + \left[y \int \cos y\, dy - \int \left\{ \frac{d}{dy}(y) \int \cos y\, dy \right\} dy \right]$$

$$= 2\left[\log x \int x\, dx - \int \left\{ \frac{d}{dx}(\log x) \int x\, dx \right\} dx \right] + \frac{x^2}{2}$$

(1)

$$\Rightarrow -\cos y + y \sin y - \int \sin y\, dy = 2\left[\frac{x^2}{2} \log x - \int \left\{ \frac{1}{x} \frac{x^2}{2} \right\} dx \right] + \frac{x^2}{2}$$

$$\Rightarrow -\cos y + y \sin y + \cos y = x^2 \log x - \int x\, dx + \frac{x^2}{2}$$

$$\Rightarrow \quad y \sin y = x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y \sin y = x^2 \log x + C \quad \text{...(i)} \quad \textbf{(1)}$$

Also, given that $y = \frac{\pi}{2}$, when $x = 1$.

On putting $y = \frac{\pi}{2}$ and $x = 1$ in Eq. (i), we get

$$\frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) = (1)^2 \log(1) + C$$

$$\Rightarrow \quad C = \frac{\pi}{2} \quad \left[\because \sin\frac{\pi}{2} = 1, \log 1 = 0\right]$$

On substituting the value of C in Eq. (i), we get

$$y \sin y = x^2 \log x + \frac{\pi}{2}$$

which is the required particular solution. **(1)**

7. Solve the following differential equation

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{2}{x^2 - 1}.$$

Delhi 2014; All India 2014C

Firstly, divide the given differential equation by $(x^2 - 1)$ to convert it into the form of linear differential equation and then solve it.

Given differential equation is

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{2}{x^2 - 1}$$

On dividing both sides by $(x^2 - 1)$, we get

$$\frac{dy}{dx} + \frac{2x}{x^2 - 1}y = \frac{2}{(x^2 - 1)^2}$$

which is a linear differential equation. **(1)**

On comparing with the form $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{2x}{x^2 - 1}, Q = \frac{2}{(x^2 - 1)^2}$$

$$\therefore \quad \text{IF} = e^{\int \frac{2x}{x^2 - 1} dx} \quad \textbf{(1)}$$

$$= e^{\log|x^2 - 1|} = x^2 - 1$$

$$\left[\begin{array}{l} \text{put } x^2 - 1 = t \Rightarrow 2x\,dx = dt \text{ in} \int \frac{2x}{x^2 - 1}dx, \text{ then} \\ \int \frac{2x}{x^2 - 1}dx = \int \frac{1}{t}dt = \log t = \log(x^2 - 1) \end{array}\right]$$

Hence, the required general solution is

$$y \cdot \text{IF} = \int Q \times \text{IF}\, dx + C$$

$$\Rightarrow y(x^2 - 1) = \int \frac{2}{(x^2 - 1)^2} \times (x^2 - 1)\, dx + C \quad \textbf{(1)}$$

$$\Rightarrow \quad y(x^2 - 1) = \int \frac{2}{x^2 - 1} dx + C$$

$$\Rightarrow \quad y(x^2-1) = \log\left|\frac{x-1}{x+1}\right| + C$$

$$\left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

which is the required differential equation. **(1)**

8. Find the particular solution of the differential equation $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}\,dy = 0$, given that $y = 1$, when $x = 0$. **Delhi 2014**

Given differential equation is

$$e^x \sqrt{1-y^2}\, dx + \frac{y}{x} dy = 0$$

$$\Rightarrow \quad e^x \sqrt{1-y^2}\, dx = \frac{-y}{x} dy$$

On separating the variables, we get

$$\frac{-y}{\sqrt{1-y^2}} dy = x\, e^x dx \qquad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{-y}{\sqrt{1-y^2}} dy = \int x\, e^x dx$$

On putting $1-y^2 = t \Rightarrow -y\, dy = \frac{dt}{2}$ in LHS, we get

$$\int \frac{1}{2\sqrt{t}} dt = \int \underset{\text{I}}{x}\, \underset{\text{II}}{e^x}\, dx$$

$$\Rightarrow \quad \frac{1}{2}[2\sqrt{t}] = x \int e^x dx - \int \left[\frac{d}{dx}(x) \int e^x dx\right] dx$$

$$\Rightarrow \quad \sqrt{1-y^2} = x\, e^x - \int e^x dx \quad [\because t = 1-y^2]$$

(1)

$$\Rightarrow \quad \sqrt{1-y^2} = x\, e^x - e^x + C \qquad \text{...(i)}$$

Also, given that $y = 1$, when $x = 0$

On putting $y = 1$ and $x = 0$ in Eq. (i), we get

$$\sqrt{1-1} = 0 - e^0 + C$$

$$\Rightarrow \quad C = 1 \qquad [\because e^0 = 1] \ \textbf{(1)}$$

On substituting the value of C in Eq. (i), we get

$$\sqrt{1-y^2} = x\, e^x - e^x + 1$$

which is the required particular solution of given differential equation. **(1)**

9. Solve the following differential equation

$\operatorname{cosec} x \log y \frac{dy}{dx} + x^2y^2 = 0.$ **Delhi 2014**

> Firstly, separate the variables, then integrate by using integration by parts.

Given differential equation is

$$\operatorname{cosec} x \log y \frac{dy}{dx} + x^2y^2 = 0 \quad \text{...(i)}$$

It can be rewritten as

$$\operatorname{cosec} x \log y \frac{dy}{dx} = -x^2y^2$$

On separating the variables, we get

$$\frac{\log y}{y^2} dy = \frac{-x^2}{\operatorname{cosec} x} dx$$

On integrating both sides, we get

$$\int \frac{\log y}{y^2} dy = -\int \frac{x^2}{\operatorname{cosec} x} dx \Rightarrow I_1 = I_2 \quad \text{...(ii)}$$

(1)

where, $I_1 = \int \frac{\log y}{y^2} dy$

Put $\log y = t \Rightarrow y = e^t$, then $\frac{dy}{y} = dt$

$$\therefore \quad I_1 = \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^{-t}}\, dt$$

$$= t \int e^{-t} dt - \int \left[\frac{d}{dt}(t) \int e^{-t} dt\right] dt$$

$$= -t\, e^{-t} - \int (-e^{-t})\, dt$$

$$= -t\, e^{-t} + \int e^{-t} dt = -t\, e^{-t} - e^{-t} + C_1$$

$$= -\frac{\log y}{y} - \frac{1}{y} + C_1 \quad \text{...(iii)} \; \textbf{(1)}$$

$$\left[\because t = \log y \text{ and } e^{-t} = \frac{1}{y}\right]$$

$\left[\quad \quad y\right]$

and $I_2 = -\int \frac{x^2}{\text{cosec} x} dx$

$= -\int \underset{\text{I}}{x^2} \underset{\text{II}}{\sin x} \, dx$

$= -x^2 \int \sin x \, dx - \int \left[\frac{d}{dx}(x^2) \int \sin x \, dx\right] dx$

$= -x^2 (-\cos x) - \int [2x(-\cos x)] \, dx$

$= x^2 \cos x + 2 \int \underset{\text{I}}{x} \underset{\text{II}}{\cos x} \, dx$

$= x^2 \cos x + 2 \left[x \int \cos x dx \right.$

$\left. - \int \left\{\frac{d}{dx}(x) \int \cos x \, dx\right\} dx \right]$

$= x^2 \cos x + 2 [x \sin x - \int \sin x \, dx]$

$= x^2 \cos x + 2x \sin x + 2 \cos x + C_2$ ...(iv)

(1)

On putting the values of I_1 and I_2 from Eqs.(iii) and (iv) in Eq. (ii), we get

$$-\frac{\log y}{y} - \frac{1}{y} + C_1 = x^2 \cos x + 2x \sin x + 2 \cos x + C_2$$

$$\Rightarrow \quad -\frac{(1 + \log y)}{y} = x^2 \cos x + 2x \sin x + 2 \cos x + C_2 - C_1$$

$$\Rightarrow \quad -\frac{(1 + \log y)}{y} = x^2 \cos x + 2x \sin x + 2 \cos x + C$$

where, $C = C_2 - C_1$

which is the required solution of given differential equation. **(1)**

10. Find the particular solution of the differential equation $x(1+y^2)\,dx - y\,(1+x^2)\,dy = 0$, given that $y = 1$, when $x = 0$. **All India 2014**

Given differential equation is

$$x(1+y^2)\,dx - y(1+x^2)\,dy = 0$$

$$\Rightarrow \quad x(1+y^2)\,dx = y(1+x^2)\,dy$$

On separating the variables, we get

$$\frac{y}{(1+y^2)}\,dy = \frac{x}{(1+x^2)}\,dx \qquad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{y}{1+y^2}\,dy = \int \frac{x}{(1+x^2)}\,dx$$

$$\Rightarrow \frac{1}{2}\log|1+y^2| = \frac{1}{2}\log|1+x^2| + C \qquad \ldots\text{(i)}$$

$$\left[\begin{array}{l} \text{let } 1+y^2 = u \Rightarrow 2y\,dy = du, \\ \text{then} \displaystyle\int \frac{y}{1+y^2}\,dy = \int \frac{1}{2u}\,du = \frac{1}{2}\log|u| \\ \text{and let } 1+x^2 = v \Rightarrow 2x\,dx = dv, \\ \text{then} \displaystyle\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{v}\,dv = \frac{1}{2}\log|v| \end{array}\right]$$

Also, given that $y = 1$, when $x = 0$. **(1)**

On substituting the values of x and y in Eq. (i), we get

$$\frac{1}{2}\log|1+(1)^2| = \frac{1}{2}\log|1+(0)^2| + C$$

$$\Rightarrow \quad \frac{1}{2}\log 2 = C \qquad [\because \log 1 = 0]$$

On putting $C = \frac{1}{2}\log 2$ in Eq. (i), we get

$$\frac{1}{2}\log|1+y^2| = \frac{1}{2}\log|1+x^2| + \frac{1}{2}\log 2$$

$$\Rightarrow \log|1+y^2| = \log|1+x^2| + \log 2 \qquad \textbf{(1)}$$

$$\Rightarrow \log|1+y^2| - \log|1+x^2| = \log 2$$

$$\Rightarrow \log\left|\frac{1+y^2}{1+x^2}\right| = \log 2 \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$\Rightarrow \qquad \frac{1+y^2}{1+x^2} = 2$$

$$\Rightarrow \qquad 1+y^2 = 2+2x^2 \quad \Rightarrow y^2 - 2x^2 - 1 = 0$$

which is the required particular solution of given differential equation. **(1)**

11. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ equation, given that $y = 0$, when $x = 0$. **All India 2014**

Given differential equation is

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{3x+4y}$$

$[\because \log m = n \Rightarrow e^n = m]$

$$\Rightarrow \quad \frac{dy}{dx} = e^{3x}\, e^{4y} \qquad \textbf{(1)}$$

On separating the variables, we get

$$\frac{1}{e^{4y}}\, dy = e^{3x} dx$$

On integrating both sides, we get

$$\int e^{-4y} dy = \int e^{3x} dx$$

$$\Rightarrow \quad \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C \qquad \text{...(i)} \ \textbf{(1)}$$

Also, given that $y = 0$, when $x = 0$.

On putting $y = 0$ and $x = 0$ in Eq. (i), we get

$$\frac{e^{-4(0)}}{-4} = \frac{e^{3(0)}}{3} + C$$

$$\Rightarrow \quad -\frac{1}{4} = \frac{1}{3} + C \qquad [\because e^{-0} = e^0 = 1]$$

$$\Rightarrow \quad C = -\frac{1}{4} - \frac{1}{3}$$

$$\therefore \quad C = \frac{-7}{12} \qquad \textbf{(1)}$$

On substituting the value of C in Eq. (i), we get

$$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} - \frac{7}{12}$$

which is the required particular solution of given differential equation. **(1)**

12. Solve the differential equation

$$(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}.$$ **All India 2014**

Given differential equation is

$$(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$$

On dividing both sides by $(1+x^2)$, we get

$$\frac{dy}{dx}+\frac{1}{(1+x^2)}y=\frac{e^{\tan^{-1}x}}{1+x^2}$$

It is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q$$

On comparing, we get

$$P=\frac{1}{1+x^2} \text{ and } Q=\frac{e^{\tan^{-1}x}}{1+x^2}$$

$$\therefore \quad \text{IF}=e^{\int P\,dx}=e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$$

$$\left[\because \int\frac{1}{1+x^2}dx=\tan^{-1}x\right]$$ **(1)**

Then, required solution is

$$(y\cdot\text{IF})=\int(Q\cdot\text{IF})\,dx+C$$

$$\therefore \quad y\,e^{\tan^{-1}x}=\int\frac{e^{\tan^{-1}x}\cdot e^{\tan^{-1}x}}{1+x^2}dx+C$$

$$\Rightarrow \quad y\,e^{\tan^{-1}x}=\int\frac{e^{2\tan^{-1}x}}{1+x^2}dx+C$$

$$\Rightarrow \quad ye^{\tan^{-1}x}=I+C \qquad \ldots\text{(i)}$$ **(1)**

where, $$I=\int\frac{e^{2\tan^{-1}x}}{1+x^2}dx$$

Put $$\tan^{-1}x=t \Rightarrow \frac{1}{1+x^2}dx=dt$$

$$\therefore \quad I=\int e^{2t}dt$$

$_{2t}$ $_{2\tan^{-1}x}$

$$\Rightarrow \qquad I = \frac{e^{\cdots}}{2} \Rightarrow I = \frac{e^{\cdots}}{2} \qquad \textbf{(1)}$$

On putting the value of I in Eq. (i), we get

$$y\, e^{\tan^{-1} x} = \frac{e^{2 \tan^{-1} x}}{2} + C$$

which is the required general solution of given differential equation. **(1)**

13. Find a particular solution of the differential equation $\frac{dy}{dx} + 2y \tan x = \sin x$, given that $y = 0$, when $x = \frac{\pi}{3}$. **Foreign 2014**

Given differential equation is

$$\frac{dy}{dx} + 2y \tan x = \sin x$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$.

On comparing, we get

$$P = 2 \tan x \text{ and } Q = \sin x$$

$\therefore \quad IF = e^{2\int \tan x\, dx} = e^{2 \log|\sec x|}$ **(1)**

$= e^{\log \sec^2 x}$ $[\because m \log n = \log n^m]$

$= \sec^2 x$ $[\because e^{\log x} = x]$

The general solution is given by

$$Y \cdot IF = \int Q \times IF\, dx + C \quad \ldots(i) \quad \textbf{(1)}$$

$$\Rightarrow \quad y \sec^2 x = \int (\sin x \cdot \sec^2 x)\, dx + C$$

$$\Rightarrow \quad y \sec^2 x = \int \sin x \cdot \frac{1}{\cos^2 x}\, dx + C$$

$$\Rightarrow \quad y \sec^2 x = \int \tan x \sec x\, dx + C$$

$$\Rightarrow \quad y \sec^2 x = \sec x + C \quad \ldots(ii)$$

Also, given that $y - 0$, when $x = \frac{\pi}{3}$. On putting

$y = 0$ and $x = \frac{\pi}{3}$ in Eq. (ii), we get

$$0 \times \sec^2 \frac{\pi}{3} = \sec \frac{\pi}{3} + C$$

$$\Rightarrow \quad 0 = 2 + C \Rightarrow C = -2 \quad \textbf{(1)}$$

On putting the value of C in Eq. (ii), we get

$$y \sec^2 x = \sec x - 2$$

$$\Rightarrow \quad y = \cos x - 2\cos^2 x$$

which is the required solution of the given differential equation. **(1)**

14. Solve the following differential equation

$$x \cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x;\ x \neq 0.$$

All India 2014C

Given differential equation is

$$x \cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x \quad \ldots(i)$$

which is a homogeneous differential equation.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$x \cos v\left[v + x\frac{dv}{dx}\right] = vx \cos v + x$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{x(v \cos v + 1)}{x \cos v} \quad \textbf{(1)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v \cos v + 1}{\cos v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v \cos v + 1 - v \cos v}{\cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1}{\cos v} \Rightarrow \cos v\, dv = \frac{dx}{x} \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \cos v\, dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \sin v = \log x + C \quad \textbf{(1)}$$

$$\Rightarrow \quad \sin\left(\frac{y}{x}\right) = \log x + C \left[\because y = vx \Rightarrow v = \frac{y}{x}\right]$$

which is the required solution of given differential equation. **(1)**

15. If $y(x)$ is a solution of the differential equation $\left(\frac{2 + \sin x}{1 + y}\right)\frac{dy}{dx} = -\cos x$ and $y(0) = 1$, then find the value of $y\left(\frac{\pi}{2}\right)$. **Delhi 2014C**

Given differential equation is

$$\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$$

$$\Rightarrow \quad \frac{1}{1+y}dy = -\frac{\cos x}{2+\sin x}dx \qquad \textbf{(1)}$$

Now, on integrating both sides, we get

$$\int \frac{1}{1+y}dy = -\int \frac{\cos x}{2+\sin x}dx$$

$$\Rightarrow \quad \log|1+y| = -\log|2+\sin x| + \log C$$

$$\left[\begin{array}{l} \text{for } \sqrt{\dfrac{\cos x}{2+\sin x}}dx, \text{let } 2+\sin x = t \\ \qquad\qquad \Rightarrow \cos x\, dx = dt, \\ \text{then } \displaystyle\int \frac{\cos x}{2+\sin x}dx = \int \frac{dt}{t} = \log t + C \\ \qquad\qquad = \log|2+\sin x| + C \end{array}\right]$$

$$\Rightarrow \quad \log(1+y) + \log(2+\sin x) = \log C$$

$$\Rightarrow \quad \log(1+y)(2+\sin x) = \log C$$

$$\Rightarrow \quad (1+y)(2+\sin x) = C \qquad \ldots\text{(i)}$$

Also, given that at $x=0, y(0)=1$

On putting $x=0$ and $y=1$ in Eq. (i), we get

$$(1+1)(2+\sin 0) = C$$

$$\Rightarrow \quad C = 4 \qquad \textbf{(1)}$$

On putting $C=4$ in Eq. (i), we get

$(1+y)(2+\sin x) = 4$

$$\Rightarrow \quad 1+y = \frac{4}{2+\sin x}$$

$$\Rightarrow \quad y = \frac{4}{2+\sin x} - 1$$

$$\Rightarrow \quad y = \frac{4-2-\sin x}{2+\sin x}$$

$$\Rightarrow \quad y = \frac{2-\sin x}{2+\sin x} \qquad \textbf{(1)}$$

Now, at $x = \frac{\pi}{2}$, $y\left(\frac{\pi}{2}\right) = \dfrac{2 - \sin\frac{\pi}{2}}{2 + \sin\frac{\pi}{2}}$

$\therefore \quad y\left(\frac{\pi}{2}\right) = \frac{1}{3}$ $\quad\left[\because \sin\frac{\pi}{2} = 1\right]$ **(1)**

16. Solve the differential equation

$x\frac{dy}{dx} + y = x \cdot \cos x + \sin x$, given $y\left(\frac{\pi}{2}\right) = 1.$

All India 2014C

Given differential equation is

$$x\frac{dy}{dx} + y = x\cos x + \sin x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{\sin x}{x}$$

[dividing on both sides by x]

which is a linear differential equation.

On comparing with the form $\frac{dy}{dx} + Py = Q$,

we get $\quad P = \frac{1}{x}$ and $Q = \cos x + \frac{\sin x}{x}$

$$\therefore \quad \text{IF} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

The general solution is given by

$$y \cdot \text{IF} = \int Q \times \text{IF}\, dx + C \quad \textbf{(1)}$$

$$\Rightarrow \quad yx = \int x\left(\cos x + \frac{\sin x}{x}\right) dx + C$$

$$\Rightarrow \quad yx = \int (x\cos x + \sin x)\, dx + C$$

$$\Rightarrow \quad xy = \int \underset{\text{I}}{x} \underset{\text{II}}{\cos x}\, dx + \int \sin x\, dx + C$$

$$\Rightarrow \quad xy = x\int \cos x\, dx - \int\left[\frac{d}{dx}(x)\int \cos x\, dx\right] dx + \int \sin x\, dx + C$$

$\Rightarrow \quad xy = x \sin x + \cos x - \cos x + C$

$\Rightarrow \quad xy = x \sin x + C$

$\Rightarrow \quad y = \sin x + C \cdot \frac{1}{x}$ ...(i) **(1)**

Also, given that at $x = \frac{\pi}{2}; y = 1$

On putting $x = \frac{\pi}{2}$ and $y = 1$ in Eq. (i), we get

$$1 = 1 + C \cdot \frac{2}{\pi} \Rightarrow C = 0$$ **(1)**

On putting the value of C in Eq. (i), we get

$$y = \sin x$$

which is the required solution of given differential equation. **(1)**

17. Solve the differential equation $\frac{dy}{dx} + y \cot x = 2 \cos x$, given that $y = 0$, when $x = \frac{\pi}{2}$. **Foreign 2014**

Given differential equation is

$$\frac{dy}{dx} + y \cot x = 2 \cos x$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \cot x$ and $Q = 2 \cos x$

$\therefore \quad IF = e^{\int P dx} = e^{\int \cot x \, dx} = e^{\log \sin x}$

$\Rightarrow \quad IF = \sin x$ **(1)**

The general solution is given by

$$Y \times IF = \int IF \times Q \, dx + C$$

$$\Rightarrow \quad y \sin x = \int 2 \sin x \cos x \, dx + C$$

$$\Rightarrow \quad y \sin x = \int \sin 2x \, dx + C$$

$$\Rightarrow \quad y \sin x = -\frac{\cos 2x}{2} + C \quad \text{...(i)}$$ **(1)**

Also, given that $y = 0$, when $x = \frac{\pi}{2}$.

On putting $x = \frac{\pi}{2}$ and $y = 0$ in Eq. (i), we get

$$0 \sin \frac{\pi}{2} = -\frac{\cos 2 \frac{\pi}{2}}{2} + C$$

$$\Rightarrow \quad C - \frac{\cos \pi}{2} = 0 \quad \Rightarrow \quad C + \frac{1}{2} = 0$$

$\therefore \quad C = -\frac{1}{2}$ **(1)**

On putting the value of C in Eq. (i), we get

$$y \sin x = -\cos \frac{2x}{2} - \frac{1}{2}$$

$\Rightarrow \; 2y \sin x + \cos 2x + 1 = 0$

which is the required solution. **(1)**

18. Solve the differential equation
$(x^2 - yx^2)\,dy + (y^2 + x^2y^2)\,dx = 0$, given that
$y = 1$, when $x = 1$. **Foreign 2014**

Direction (Q. Nos. 19-22) *Solve the following differential equations.*

Given differential equation is

$(x^2 - yx^2)dy + (y^2 + x^2y^2)\,dx = 0$

On dividing both sides by dx, we get

$$(x^2 - yx^2)\frac{dy}{dx} + (y^2 + x^2y^2) = 0$$

$$\Rightarrow \quad x^2(1 - y)\frac{dy}{dx} + y^2(1 + x^2) = 0$$

$$\Rightarrow \quad -x^2(1 - y)\frac{dy}{dx} = y^2(1 + x^2)$$

$$\Rightarrow \quad x^2 (y-1) \frac{dy}{dx} = y^2 (1+x^2)$$

$$\Rightarrow \quad \frac{y-1}{y^2} dy = \frac{1+x^2}{x^2} dx \qquad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{y-1}{y^2} dy = \int \frac{1+x^2}{x^2} dx$$

$$\Rightarrow \frac{1}{2} \int \frac{2y}{y^2} dy - \int \frac{1}{y^2} dy = \int \frac{1}{x^2} dx + \int 1 \cdot dx \qquad \textbf{(1)}$$

On putting $y^2 = t \Rightarrow 2y\, dy = dt$ in first integral, we get

$$\frac{1}{2} \int \frac{dt}{t} + \frac{1}{y} = -\frac{1}{x} + x$$

$$\Rightarrow \quad \frac{1}{2} \log |y^2| + \frac{1}{y} = -\frac{1}{x} + x + C \qquad \ldots(\text{i})$$

$$[\because t = y^2]$$

Also, given that $y = 1$, when $x = 1$.

On putting $y = 1$ and $x = 1$ in Eq.(i), we get

$$\frac{1}{2} \log |1| + \frac{1}{1} = \frac{-1}{1} + 1 + C$$

$$\Rightarrow \quad \frac{1}{2} \log |1| + 1 = -1 + 1 + C$$

$$\Rightarrow \quad C = 1 \qquad [\because \log 1 = 0] \ \textbf{(1)}$$

On putting the value of C in Eq. (i), we get

$$\frac{1}{2} \log |y^2| + \frac{1}{y} = -\frac{1}{x} + x + 1$$

which is the required solution. **(1)**

19. $\frac{dy}{dx} + y \sec x = \tan x$ All India 2012C; Delhi 2008C

Given differential equation is

$$\frac{dy}{dx} + y \sec x = \tan x \quad \ldots(i)$$

which is a linear differential equation of first order and is of the form

$$\frac{dy}{dx} + Py = Q \quad \ldots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \sec x \text{ and } Q = \tan x \quad \textbf{(1)}$$

$$\therefore \quad IF = e^{\int \sec x\, dx} = e^{\log|\sec x + \tan x|}$$

$$[\because \int \sec x\, dx = \log|\sec x + \tan x|]$$

$$\Rightarrow \quad IF = \sec x + \tan x \quad \textbf{(1)}$$

The general solution is

$$y \times IF = \int Q \cdot IF\, dx + C$$

$$y(\sec x + \tan x) = \int \tan x \cdot (\sec x + \tan x)\, dx$$

$$\Rightarrow y(\sec x + \tan x) = \int \sec x \tan x\, dx + \int \tan^2 x\, dx$$

$$\Rightarrow y(\sec x + \tan x) = \sec x + \int (\sec^2 x - 1)\, dx \quad \textbf{(1)}$$

$$\Rightarrow \quad y(\sec x + \tan x) = (\sec x + \tan x) - x + C$$

$$[\because \int \sec^2 x\, dx = \tan x + C]$$

On dividing both sides by $(\sec x + \tan x)$, we get the required solution as

$$y = 1 - \frac{x}{\sec x + \tan x} + \frac{C}{\sec x + \tan x} \quad \textbf{(1)}$$

20. $2x^2 \frac{dy}{dx} - 2xy + y^2 = 0$ Delhi 2012

Given differential equation is

$$2x^2 \frac{dy}{dx} - 2xy + y^2 = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2xy - y^2}{2x^2} \qquad ...(i) \textbf{ (1)}$$

which is a homogeneous differential equation.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{2vx^2 - v^2x^2}{2x^2} \qquad \textbf{(1)}$$

$$\Rightarrow \qquad v + x\frac{dv}{dx} = \frac{2v - v^2}{2}$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \frac{2v - v^2}{2} - v$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \frac{2v - v^2 - 2v}{2}$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \frac{-v^2}{2}$$

$$\Rightarrow \qquad \frac{2dv}{v^2} = -\frac{1}{x}dx \qquad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{2dv}{v^2} = \int \frac{-dx}{x} + C$$

$$\Rightarrow \quad 2\int v^{-2}dv = -\log|x| + C$$

$$\Rightarrow \qquad \frac{2v^{-1}}{-1} = -\log|x| + C$$

$$\Rightarrow \qquad \frac{-2}{v} = -\log|x| + C$$

$$\Rightarrow \qquad \frac{-2x}{y} = -\log|x| + C$$

$$\left[\because y = vx \Rightarrow v = \frac{y}{x}\right]$$

$$\Rightarrow \qquad -2x = y(-\log|x| + C)$$

$$\Rightarrow \qquad y = \frac{-2x}{-\log|x| + C}$$

which is the required solution. **(1)**

21. $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$, given that $y = 1$, when $x = 0$. **Delhi 2012**

Given differential equation is

$$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2 \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + x^2) + y^2(1 + x^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + x^2)(1 + y^2)$$

$$\Rightarrow \quad \frac{dy}{1 + y^2} = (1 + x^2)\,dx \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{dy}{1 + y^2} = \int (1 + x^2)\,dx$$

$$\Rightarrow \quad \tan^{-1} y = x + \frac{x^3}{3} + C \quad \ldots\text{(i)}$$

Also, given that $y = 0$, when $x = 2$.

On putting $x = 0$ and $y = 1$ in Eq. (i), we get

$$\tan^{-1} 1 = C$$

$$\Rightarrow \quad \tan^{-1}(\tan \pi/4) = C \quad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$\Rightarrow \quad C = \pi/4 \quad \textbf{(1)}$$

On putting the value of C in Eq. (i), we get

$$\tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}$$

$$\Rightarrow \quad y = \tan\left(x + \frac{x^3}{3} + \frac{\pi}{4}\right)$$

which is the required solution. **(1)**

22. $x(x^2 - 1)\frac{dy}{dx} = 1$, $y = 0$, when $x = 2$. **All India 2012**

Given differential equation is

$$x(x^2-1)\frac{dy}{dx}=1$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x(x^2-1)}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x(x-1)(x+1)}$$

$$[\because a^2-b^2=(a-b)(a+b)]$$

$$\Rightarrow \quad dy=\frac{dx}{x(x-1)(x+1)}$$

On integrating both sides, we get

$$\int dy=\int \frac{dx}{x(x-1)(x+1)}+C$$

$$\Rightarrow \quad y=I+C \quad \ldots(i)$$

where, $I=\int \frac{dx}{x(x-1)(x+1)}$ **(1)**

Let $\frac{1}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$

$\Rightarrow \quad 1=A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$

On comparing coefficients of x^2, x and constant terms from both sides, we get

$$A+B+C=0 \quad \ldots(ii)$$

$$B-C=0 \quad \ldots(iii)$$

and $-A=1$

$\Rightarrow \quad A=-1$

On putting $A=-1$ in Eq. (ii), we get

$$B+C=1 \quad \ldots(iv)$$

Now, on adding Eqs. (iii) and (iv), we get

$$2B=1 \quad \Rightarrow \quad B=\frac{1}{2}$$

On putting $B=\frac{1}{2}$ in Eq. (iii), we get

$$\frac{1}{-}-C=0 \quad \Rightarrow \quad C=\frac{1}{-}$$

2 2

$\therefore \quad A = -1, B = \frac{1}{2}$ and $C = \frac{1}{2}$,

then $\frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}$ **(1)**

On integrating both sides w.r.t. x, we get

$$I = \int \frac{1}{x(x-1)(x+1)} dx = \int \frac{-1}{x} dx + \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1}$$

$$\Rightarrow \quad I = -\log|x| + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1|$$

On putting the value of I in Eq. (i), we get

$$y = -\log|x| + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| + C \quad \ldots (v)$$

Also, given that $y = 0$, when $x = 2$.

On putting $y = 0$ and $x = 2$ in Eq. (v), we get

$$0 = -\log 2 + \frac{1}{2}\log 1 + \frac{1}{2}\log 3 + C$$

$$\Rightarrow \quad C = \log 2 - \frac{1}{2}\log 1 - \frac{1}{2}\log 3$$

$$\Rightarrow \quad C = \log 2 - \log\sqrt{3} \qquad [\because \log 1 = 0]$$

$$\Rightarrow \quad C = \log\frac{2}{\sqrt{3}}$$ **(1)**

On putting the value of C in Eq. (v), we get

$$y = -\log|x| + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| + \log\frac{2}{\sqrt{3}}$$ **(1)**

which is the required solution.

23. Solve the following differential equation

$\frac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x$, given that $y = 0$,

when $x = \frac{\pi}{2}$. Delhi 2012C; Foreign 2011

Given differential equation is

$$\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$$

which is a linear differential equation.
On comparing with general form of linear differential equation of 1st order

$$\frac{dy}{dx} + Py = Q \text{, we get}$$

$$P = \cot x \text{ and } Q = 4x \operatorname{cosec} x \quad \textbf{(1)}$$

$$\therefore \quad \text{IF} = e^{\int P dx} = e^{\int \cot x \, dx}$$

$$= e^{\log \sin x} = \sin x \quad [\because e^{\log x} = x]$$

$$\Rightarrow \quad \text{IF} = \sin x \quad \textbf{(1)}$$

Now, solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C$$

On putting IF $= \sin x$ and $Q = 4x \operatorname{cosec} x$, we get

$$y \times \sin x = \int 4x \operatorname{cosec} x \cdot \sin x \, dx + C$$

$$\Rightarrow \quad y \sin x = \int 4x \cdot \frac{1}{\sin x} \cdot \sin x \, dx + C$$

$$\Rightarrow \quad y \sin x = \int 4x \, dx + C$$

$$\Rightarrow \quad y \sin x = 2x^2 + C \quad \text{...(i)} \ \textbf{(1)}$$

Also, given that $y = 0$, when $x = \frac{\pi}{2}$.

On putting $y = 0$ and $x = \frac{\pi}{2}$ in Eq. (i), we get

$$0 = 2 \times \frac{\pi^2}{4} + C \Rightarrow C = \frac{-\pi^2}{2}$$

On putting $C = -\frac{\pi^2}{2}$ in Eq. (i), we get

$$y \sin x = 2x^2 - \frac{\pi^2}{2}$$

$$\Rightarrow \quad y = 2x^2 \operatorname{cosec} x - \frac{\pi^2}{2} \operatorname{cosec} x \quad \textbf{(1)}$$

which is the required solution.

24. Solve the following differential equation $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx$, where $x \neq 0$.

All India 2012C, 2011

Given differential equation is

$$(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx \qquad [\because x \neq 0]$$

$$\Rightarrow \quad (1 + x^2)\, dy = (\cot x - 2xy)\, dx$$

On dividing both sides by $1 + x^2$, we get

$$dy = \frac{\cot x - 2xy}{1 + x^2} dx$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2xy}{1 + x^2} = \frac{\cot x}{1 + x^2} \quad \text{...(i)} \quad \textbf{(1)}$$

which is a linear differential equation of 1st order and is of the form

$$\frac{dy}{dx} + Py = Q \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{2x}{1 + x^2} \text{ and } Q = \frac{\cot x}{1 + x^2}$$

$$\therefore \quad \text{IF} = e^{\int \frac{2x}{1 + x^2} dx}$$

$$= e^{\log|1 + x^2|} = 1 + x^2 \quad \textbf{(1)}$$

$$\left[\text{for} \int \frac{2x}{1 + x^2} dx, \text{put } 1 + x^2 = t \Rightarrow 2x\, dx = dt\right]$$

$$\left[\int \frac{dt}{t} = \log|t| = \log|1+x^2| + C \right]$$

Now, solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C$$

$$\therefore \quad y(1+x^2) = \int \frac{\cot x}{1+x^2} \times (1+x^2)\, dx + C$$

$$\Rightarrow \quad y(1+x^2) = \int \cot x\, dx + C \qquad \textbf{(1)}$$

$$\Rightarrow \quad y(1+x^2) = \log|\sin x| + C$$

$$\left[\because \int \cot x\, dx = \log|\sin x| + C\right]$$

On dividing both sides by $1+x^2$, we get

$$y = \frac{\log|\sin x|}{1+x^2} + \frac{C}{1+x^2}$$

which is the required solution. **(1)**

25. Find the particular solution of the differential equation

$(1+e^{2x})dy + (1+y^2)e^x dx = 0$, given that $y = 1$, when $x = 0$. **Foreign 2011; All India 2008C**

Given differential equation is

$(1+e^{2x})\, dy + (1+y^2)e^x dx = 0$

Above equation may be written as

$$\frac{dy}{1+y^2} = \frac{-e^x}{1+e^{2x}} dx \qquad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{dx}{1+y^2} = -\int \frac{e^x}{1+e^{2x}} dx$$

On putting $e^x = t \Rightarrow e^x dx = dt$ in RHS, we get

$$\tan^{-1} y = -\int \frac{1}{1+t^2} dt$$

$$\Rightarrow \quad \tan^{-1} y = -\tan^{-1} t + C$$

$$\Rightarrow \quad \tan^{-1} y = -\tan^{-1}(e^x) + C \quad \ldots(i)$$

$[\because t = e^x]$ **(1½)**

Now, given that $y = 1$, when $x = 0$.

On putting above values in Eq. (i), we get

$$\tan^{-1} 1 = -\tan^{-1}(e^0) + C$$

$$\Rightarrow \quad \tan^{-1}\left(\tan \frac{\pi}{4}\right) = -\tan^{-1} 1 + C \quad [\because e^0 = 1]$$

$$\Rightarrow \quad \frac{\pi}{4} = -\tan^{-1}\left(\tan \frac{\pi}{4}\right) + C$$

$$\Rightarrow \quad \frac{\pi}{4} = -\frac{\pi}{4} + C$$

$$\Rightarrow \quad C = \frac{\pi}{4} + \frac{\pi}{4} \Rightarrow C = \frac{\pi}{2}$$

On putting $C = \frac{\pi}{2}$ in Eq. (i), we get

$$\tan^{-1} y = -\tan^{-1} e^x + \frac{\pi}{2}$$

$$\Rightarrow \quad y = \tan\left[\frac{\pi}{2} - \tan^{-1}(e^x)\right] = \cot\left[\tan^{-1}(e^x)\right]$$

$$= \cot\left[\cot^{-1}\left(\frac{1}{e^x}\right)\right] \quad \left[\because \tan^{-1} x = \cot^{-1} \frac{1}{x}\right]$$

$$\Rightarrow \quad y = \frac{1}{e^x}$$

which is the required solution. **(1½)**

26. Solve the following differential equation

$(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$, given that $y=0$,

when $x=1$. **Foreign 2011**

Given differential equation is

$$(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$$

On dividing both sides by $(1+x^2)$, we get

$$\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{1}{(1+x^2)^2} \quad \dots(i)$$

which is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P=\frac{2x}{1+x^2} \text{ and } Q=\frac{1}{(1+x^2)^2} \quad \textbf{(1)}$$

$$\therefore \quad IF = e^{\int \frac{2x}{1+x^2}dx} = e^{\log|1+x^2|} \quad \textbf{(1)}$$

$$\Rightarrow \quad IF = 1+x^2 \quad [\because e^{\log x} = x]$$

$$\left[\because \int \frac{2x}{1+x^2}dx, \text{ put } 1+x^2+t \Rightarrow 2x\,dx = dt \quad \therefore \int \frac{dt}{t} = \log|t| = \log|1+x^2|\right]$$

Now, solution of linear equation is given by

$$y \times IF = \int (Q \times IF)\,dx + C \quad \text{...(iii)}$$

$$\therefore \quad y(1+x^2) = \int \frac{1}{(1+x^2)^2} \times (1+x^2)\,dx + C$$

$$\Rightarrow \quad y(1+x^2) = \int \frac{1}{1+x^2}dx + C$$

$$\Rightarrow \quad y(1+x^2) = \tan^{-1}x + C \quad \text{...(iv)} \ \textbf{(1)}$$

$$\left[\because \int \frac{1}{1+x^2}dx = \tan^{-1}x + C\right]$$

Also, given that $y=0$, when $x=1$.

On putting $y=0$ and $x=1$ in Eq. (iv), we get

$$0 = \tan^{-1}1 + C$$

$$\Rightarrow \quad 0 = \tan^{-1}\left(\tan\frac{\pi}{4}\right) + C \quad \left[\because 1 = \tan\frac{\pi}{4}\right]$$

$$\Rightarrow \quad 0 = \frac{\pi}{4} + C \quad \Rightarrow \quad C = \frac{-\pi}{4}$$

On putting $C = \frac{-\pi}{4}$ in Eq. (iv), we get

$$y(1+x^2) = \tan^{-1}x - \frac{\pi}{4}$$

$$\Rightarrow \quad y = \frac{\tan^{-1}x}{1+x^2} - \frac{\pi}{4(1+x^2)} \quad \textbf{(1)}$$

which is the required solution.

27. Solve the following differential equation
$x dy - y dx = \sqrt{x^2 + y^2} dx.$ All India 2011

Given differential equation is

$$x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$$

$$\Rightarrow \quad (y + \sqrt{x^2 + y^2})dx = x\,dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \quad \text{...(i)} \quad \textbf{(1)}$$

which is a homogeneous differential equation because each term have same degree.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ **(1)**

in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x} = \frac{vx + x\sqrt{1 + v^2}}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v + \sqrt{1 + v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1 + v^2} \quad \Rightarrow \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \log|v + \sqrt{1 + v^2}| = \log|x| + C$$

$$\left[\because \int \frac{dx}{\sqrt{a^2 + x^2}} = \log|x + \sqrt{x^2 + a^2}| \right.$$

$$\left. \text{and } \int \frac{dx}{x} = \log|x| + C \right] \quad \textbf{(1)}$$

$$\Rightarrow \quad \log\left|\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right| = \log|x| + C \quad \left[\because y = vx \; \frac{y}{x} = v\right]$$

$$\left| x \quad y \quad x^2 \right| \qquad \left[\therefore v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \log\left|\frac{y + \sqrt{x^2 + y^2}}{x}\right| - \log|x| = C$$

$$\Rightarrow \quad \log\left|\frac{\dfrac{y + \sqrt{x^2 + y^2}}{x}}{x}\right| = C$$

$$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \quad \frac{y + \sqrt{x^2 + y^2}}{x^2} = e^C \quad \left[\begin{array}{l}\because \text{if } \log y = x, \\ \text{then } y = e^x\end{array}\right]$$

$$\Rightarrow \quad y + \sqrt{x^2 + y^2} = x^2 \cdot e^C$$

$\therefore y + \sqrt{x^2 + y^2} = Ax^2$ [where, $A = e^C$]**(1)**

which is the required solution.

28. Solve the following differential equation

$(y + 3x^2)\dfrac{dx}{dy} = x.$ All India 2011

Given differential equation is

$$(y + 3x^2)\frac{dx}{dy} = x \Rightarrow \frac{dy}{dx} = \frac{y}{x} + 3x$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 3x \qquad \text{...(i) } \mathbf{(1)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{-1}{x} \text{ and } Q = 3x \quad \textbf{(1)}$$

$$\therefore \quad \text{IF} = e^{\int -\frac{1}{x} dx} = e^{-\log|x|} = e^{\log x^{-1}} = x^{-1}$$

$$\Rightarrow \quad \text{IF} = x^{-1} = \frac{1}{x}$$

Now, solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C$$

$$\therefore \quad y \times \frac{1}{x} = \int 3x \times \frac{1}{x} dx \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{y}{x} = \int 3\, dx \Rightarrow \frac{y}{x} = 3x + C$$

$$\Rightarrow \quad y = 3x^2 + Cx$$

which is the required solution. **(1)**

29. Solve the following differential equation $x dy - (y + 2x^2)\, dx = 0$. All India 2011

Given differential equation is

$x\,dy - (y + 2x^2)\,dx = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y + 2x^2}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 2x \qquad \text{...(i)} \quad \textbf{(1)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{-1}{x} \text{ and } Q = 2x \qquad \textbf{(1)}$$

$$\therefore \quad \text{IF} = e^{\int -\frac{1}{x}dx} = e^{-\log|x|} = x^{-1} = \frac{1}{x} \qquad \textbf{(1)}$$

Now, solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C$$

$$\therefore \quad \frac{y}{x} = \int \left(2x \times \frac{1}{x}\right) dx + C$$

$$\Rightarrow \quad \frac{y}{x} = \int 2\,dx + C \Rightarrow \frac{y}{x} = 2x + C$$

$$\Rightarrow \quad y = 2x^2 + Cx$$

which is the required solution. **(1)**

30. Solve the following differential equation $x\,dy + (y - x^3)\,dx = 0$. All India 2011

Given differential equation is

$x\,dy + (y - x^3)\,dx = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^3 - y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = x^2 - \frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = x^2 \quad \text{...(i)} \quad \textbf{(1)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{1}{x} \text{ and } Q = x^2 \quad \textbf{(1)}$$

$$\therefore \quad \text{IF} = e^{\int \frac{1}{x} dx} = e^{\log|x|} = x \quad \textbf{(1)}$$

Solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C$$

$$\therefore \quad y \times x = \int x^2 \times x\,dx + C$$

$$\Rightarrow \quad yx = \int x^3 dx + C$$

$$\Rightarrow \quad yx = \frac{x^4}{4} + C \Rightarrow y = \frac{x^3}{4} + \frac{C}{x}$$

which is the required solution. **(1)**

31. Solve the following differential equation $e^x \tan y\,dx + (1 - e^x) \sec^2 y\,dy = 0.$ **Delhi 2011**

Given differential equation is

$$e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$$

$$\Rightarrow \quad \frac{e^x}{e^x - 1} dx = \frac{\sec^2 y}{\tan y} dy \qquad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{e^x}{e^x - 1} dx = \int \frac{\sec^2 y}{\tan y} dy$$

On putting $e^x - 1 = t$ and $\tan y = z$

$\Rightarrow \quad e^x\, dx = dt$ and $\sec^2 y\, dy = dz$

$$\therefore \quad \int \frac{dt}{t} = \int \frac{dz}{z} \qquad \textbf{(1)}$$

$$\Rightarrow \quad \log|t| = \log|z| + \log C \left[\because \int \frac{1}{x} dx = \log|x|\right]$$

$$\Rightarrow \quad \log|e^x - 1| = \log|\tan y| + \log C$$

$$\Rightarrow \quad \log|e^x - 1| = \log|C \cdot \tan y|$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad e^x - 1 = C \tan y \qquad \textbf{(1)}$$

$$\Rightarrow \quad \tan y = \frac{e^x - 1}{C} \Rightarrow y = \tan^{-1}\left(\frac{e^x - 1}{C}\right)$$

which is the required solution. **(1)**

32. Solve the following differential equation

$(1 + y^2)(1 + \log x)\, dx + x dy = 0.$ **Delhi 2011**

Given differential equation is

$$(1+y^2)(1+\log x)\,dx + x\,dy = 0$$

$$\Rightarrow \quad \frac{1+\log x}{x}\,dx = \frac{-dy}{1+y^2} \qquad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{1+\log x}{x}\,dx = -\int \frac{dy}{1+y^2}$$

$$\Rightarrow \quad \int \frac{1}{x}\,dx + \int \frac{\log x}{x}\,dx = -\int \frac{dy}{1+y^2} \qquad \textbf{(1½)}$$

$$\Rightarrow \quad \log|x| + \frac{(\log x)^2}{2} + C = -\tan^{-1} y$$

$$\left[\begin{array}{l}\text{for} \int \frac{\log x}{x}\,dx \Rightarrow \text{put } \log x = t \Rightarrow \frac{1}{x}\,dx = dt \\ \qquad \therefore \int t\,dt = \frac{t^2}{2} + C = \frac{(\log x)^2}{2} + C\end{array}\right]$$

$$\Rightarrow \quad \tan^{-1} y = -\left[\log|x| + \frac{(\log x)^2}{2} + C\right]$$

$$\Rightarrow \quad y = \tan\left[-\log|x| - \frac{(\log x)^2}{2} - C\right]$$

which is the required solution. **(1½)**

33. Solve the following differential equation

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0.$$ **Delhi 2011C**

Given differential equation is

$$\left[x \sin^2\left(\frac{y}{x}\right) - y\right] dx + x\,dy = 0$$

which is a homogeneous differential equation.

This equation can be written as

$$\left[x \sin^2\left(\frac{y}{x}\right) - y\right] dx = -x\,dy$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y - x \sin^2\left(\frac{y}{x}\right)}{x} \qquad \ldots(i)$$

On putting $y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get **(1)**

$$v + x\frac{dv}{dx} = \frac{vx - x \sin^2\left(\frac{vx}{x}\right)}{x} = v - \sin^2 v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \quad \frac{dv}{\sin^2 v} = -\frac{dx}{x} \qquad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{dv}{\sin^2 v} = -\int \frac{dx}{x}$$

$$\Rightarrow \int \operatorname{cosec}^2 v \, dv = -\int \frac{dx}{x} \left[\because \frac{1}{\sin^2 v} = \operatorname{cosec}^2 v\right]$$

$$\Rightarrow \quad -\cot v = -\log x + C$$

$$[\because \int \operatorname{cosec}^2 v \, dv = -\cot v + C] \quad \textbf{(1)}$$

$$\Rightarrow \quad -\cot\left(\frac{y}{x}\right) = -\log x + C \left[\because y = vx \therefore v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \cot\left(\frac{y}{x}\right) = \log x - C$$

$$\Rightarrow \quad \frac{y}{x} = \cot^{-1}(\log x - C) \qquad \textbf{(1)}$$

$$\Rightarrow y = x \cdot \cot^{-1}(\log x - C)$$

which is the required solution.

34. Solve the following differential equation

$x\frac{dy}{dx} + y - x + xy\cot x = 0, x \neq 0.$ **Delhi 2011C**

Given differential equation is

$$x\frac{dy}{dx}+y-x+xy\cot x=0,\ x\neq 0$$

Above equation can be written as

$$x\frac{dy}{dx}+y(1+x\cot x)=x$$

On dividing both sides by x, we get

$$\frac{dy}{dx}+y\left(\frac{1+x\cot x}{x}\right)=1$$

$$\Rightarrow \quad \frac{dy}{dx}+y\left(\frac{1}{x}+\cot x\right)=1 \qquad \ldots(i) \quad \mathbf{(1)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q \qquad \ldots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P=\frac{1}{x}+\cot x \text{ and } Q=1$$

$$\therefore \quad \text{IF}=e^{\int P dx}=e^{\int\left(\frac{1}{x}+\cot x\right)dx}=e^{\log|x|+\log\sin x}$$

$$\left[\because \int\frac{1}{x}dx=\log|x| \text{ and } \int\cot x\,dx=\log|\sin x|\right]$$

$$= e^{\log|x \sin x|}$$

$[\because \log m + \log n = \log mn]$

$\Rightarrow \quad IF = x \sin x$ **(1/2)**

$y \times IF = \int (Q \times IF)\, dx + C$ **(1/2)**

$$y \times x \sin x = \int 1 \times x \sin x\, dx + C$$

$$\Rightarrow \quad y\, x \sin x = \int \underset{I}{x} \underset{II}{\sin x}\, dx + C$$

$$\Rightarrow \quad y\, x \sin x = x \int \sin x\, dx$$

$$- \int \left(\frac{d}{dx}(x) \cdot \int \sin x\, dx \right) dx + C$$

[using integration by parts in $\int x \sin x\, dx$]

$\Rightarrow y\, x \sin x = -x \cos x - \int 1(-\cos x)\, dx + C$ **(1)**

$$\Rightarrow \quad y\, x \sin x = -x \cos x + \int \cos x\, dx + C$$

$$\Rightarrow \quad y\, x \sin x = -x \cos x + \sin x + C$$

On dividing both sides by $x \sin x$, we get

$$y = \frac{-x \cos x + \sin x + C}{x \sin x}$$

$$\Rightarrow \quad y = -\cot x + \frac{1}{x} + \frac{C}{x \sin x}$$

which is the required solution. **(1)**

35. Show that the following differential equation is homogeneous and then solve it.

$$y\, dx + x \log\left(\frac{y}{x}\right) dy - 2x\, dy = 0$$

HOTS; All India 2011C

Let the value of $\frac{dy}{dx}$ be $f(x, y)$. Now, put $x = \lambda x$ and $y = \lambda y$ and verify whether $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ $n \in Z$. If above equation is satisfied, then given equation is said to be homogeneous equation. Then, we use the substitution $y = vx$ to solve the equation.

Given differential equation is

$$y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$$

$$\Rightarrow \quad y\,dx = \left[2x - x\log\left(\frac{y}{x}\right)\right]dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} \quad \ldots(i) \quad \textbf{(1/2)}$$

Now, let $f(x, y) = \dfrac{y}{2x - x\log\left(\frac{y}{x}\right)}$

On replace x by λx and y by λy both sides, we get

$$f(\lambda x, \lambda y) = \frac{\lambda y}{2\lambda x - \lambda x\log\left(\frac{\lambda y}{\lambda x}\right)}$$

$$= \frac{\lambda y}{\lambda\left[2x - x\log\left(\frac{y}{x}\right)\right]}$$

$$\Rightarrow \quad f(\lambda x, \lambda y) = \lambda^0 \frac{y}{2x - x\log\left(\frac{y}{x}\right)} = \lambda^0 f(x, y)$$

So, given differential equation is homogeneous. **(1/2)**

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log\left(\frac{vx}{x}\right)} = \frac{v}{2 - \log v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{2 - \log v} - v = \frac{v - 2v + v\log v}{2 - \log v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v + v\log v}{2 - \log v}$$

$$\Rightarrow \quad \frac{2 - \log v}{v\log v - v}\,dv = \frac{dx}{x} \qquad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{2 - \log v}{v(\log v - 1)}\,dv = \int \frac{dx}{x}$$

On putting $\log v = t \Rightarrow \frac{1}{v}dv = dt$

Then, $\int \frac{2 - t}{t - 1}\,dt = \log|x| + C$

$$\Rightarrow \quad \int\left(\frac{1}{t - 1} - 1\right)dt = \log|x| + C \qquad \textbf{(1)}$$

$$\left[\begin{array}{l} \because t-1\,\overline{)\,2-t\,(}\,-1 \\ \qquad\quad 1-t \\ \qquad\quad -\;\;+ \\ \hline \qquad\quad 1 \\ \text{and use} \int\left(\frac{R}{D} + Q\right)dt \end{array}\right]$$

$$\Rightarrow \quad \log|t - 1| - t = \log|x| + C$$

$$\Rightarrow \quad \log|\log v - 1| - \log v = \log|x| + C$$

$$\Rightarrow \quad \log\left|\frac{\log v - 1}{v}\right| = \log|x| + C$$

$$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \log\left|\frac{\log v - 1}{v}\right| - \log|x| = C$$

$$\Rightarrow \log\left|\frac{\log v - 1}{vx}\right| = C$$

$$\therefore \log\left|\frac{\log\frac{y}{x} - 1}{y}\right| = C \qquad \left[\because y = vx \Rightarrow v = \frac{y}{x}\right]$$

which is the required solution. **(1)**

36. Solve the following differential equation

$$\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y - \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\frac{dy}{dx} = 0.$$

All India 2010C

Firstly, convert the given differential equation in homogeneous and then put $y = vx$.

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Further, separate the variables and integrate it.

Given differential equation is

$$\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)\cdot y$$

$$-\left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)\cdot x\frac{dy}{dx} = 0$$

which is a homogeneous differential equation. It can be written as

$$\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)\cdot y$$

$$= \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)\cdot x\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left[x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right]\cdot y}{\left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)\cdot x} \quad \text{...(i)}$$

On putting $\quad y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \text{ in Eq. (i), we get} \quad \textbf{(1)}$$

$$v + x\frac{dv}{dx} = \frac{(x\cos v + vx\sin v)\cdot vx}{(vx\sin v - x\cos v)\cdot x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

(1)

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \quad \frac{v\sin v - \cos v}{v\cos v}dv = 2\frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{v\sin v - \cos v}{v\cos v}dv = 2\int \frac{dx}{x}$$

$$\Rightarrow \quad \int\left(\frac{v\sin v}{v\cos v} - \frac{\cos v}{v\cos v}\right)dv = 2\int\frac{dx}{x}$$

$$\Rightarrow \quad \int\left(\tan v - \frac{1}{v}\right)dv = 2\int\frac{dx}{x}$$

$$\Rightarrow \quad \log|\sec v| - \log|v| = 2\log|x| + C$$ **(1)**

$$\left[\because \int \tan v\,dv = \log|\sec v| \text{ and } \int\frac{1}{x}dx = \log|x|\right]$$

$$\Rightarrow \quad \log|\sec v| - \log|v| - 2\log|x| = C$$

$$\Rightarrow \quad \log|\sec v| - [\log|v| + \log|x|^2 = C$$

$$[\because \log m^n = n\log m]$$

$$\Rightarrow \log|\sec v| - \log|vx^2| = C$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad \log\left|\frac{\sec v}{vx^2}\right| = C$$

$$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \quad \log\left|\frac{\sec\frac{y}{x}}{\frac{y}{x}\cdot x^2}\right| = C \qquad \left[\begin{array}{l}\because y = vx \\ \therefore v = \dfrac{y}{x}\end{array}\right]$$

$$\Rightarrow \quad \log\left|\frac{\sec\frac{y}{x}}{xy}\right| = C$$

which is the required solution **(1)**

which is the required solution. (1)

37. Solve the following differential equation

$$xy \log\left(\frac{y}{x}\right) dx + \left[y^2 - x^2 \log\left(\frac{y}{x}\right)\right] dy = 0.$$

Delhi 2010C

Given differential equation is

$$xy \log\left(\frac{y}{x}\right) dx + \left[y^2 - x^2 \log\left(\frac{y}{x}\right)\right] dy = 0$$

which is a homogeneous differential equation. This equation can be written as

$$xy \log\left(\frac{y}{x}\right) dx = \left[x^2 \log\left(\frac{y}{x}\right) - y^2\right] dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xy \log\left(\frac{y}{x}\right)}{x^2 \log\left(\frac{y}{x}\right) - y^2} \quad \ldots(i)$$

Now, put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ **(1)**

in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{vx^2 \log\left(\frac{vx}{x}\right)}{x^2 \log\left(\frac{vx}{x}\right) - v^2x^2} = \frac{v \log v}{\log v - v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v \log v}{\log v - v^2} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v \log v - v \log v + v^3}{\log v - v^2} = \frac{v^3}{\log v - v^2}$$

$$\Rightarrow \quad \frac{\log v - v^2}{v^3} dv = \frac{dx}{x} \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{\log v - v^2}{v^3} dv = \int \frac{dx}{x}$$

$$\int \frac{\log v}{\ldots} \quad \int \frac{1}{\ldots} \quad \int \frac{dx}{\ldots}$$

$$\Rightarrow \quad \int \frac{\log v}{v^3}\,dv - \int \frac{1}{v}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \int \underset{\text{I}}{v^{-3}} \underset{\text{II}}{\log v}\,dv - \log|v| = \log|x| + C$$

Using integration by parts, we get

$$\log v \int v^{-3}dv - \int\left[\frac{d}{dv}(\log v)\cdot\int v^{-3}dv\right]dv$$

$$= \log|v| + \log|x| + C$$

$$\Rightarrow \frac{v^{-2}}{-2}\log v - \int \frac{1}{v}\frac{v^{-2}}{(-2)}dv = \log|v| + \log|x| + C$$

$$\Rightarrow \frac{-1}{2v^2}\log v + \frac{1}{2}\int v^{-3}dv = \log|v| + \log|x| + C$$

$$\Rightarrow \quad \frac{-1}{2v^2}\log v + \frac{1}{2}\cdot\frac{v^{-2}}{(-2)} = \log|v| + \log|x| + C$$

$$\left[\because \int x^n\,dx = \frac{x^{n+1}}{n+1} + C\right]$$

$$\Rightarrow \quad \frac{-1}{2v^2}\log v - \frac{1}{4v^2} = \log|vx| + C \qquad \textbf{(1)}$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad \frac{-1}{2}\cdot\frac{x^2}{y^2}\log\left(\frac{y}{x}\right)-\frac{1}{4}\cdot\frac{x^2}{y^2}=\log\left|\frac{y}{x}\cdot x\right|+C$$

$$\left[\because y = vx \Rightarrow v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \frac{-x^2}{2y^2}\log\left(\frac{y}{x}\right)-\frac{x^2}{4y^2}=\log|y|+C$$

$$\Rightarrow \quad \frac{-x^2}{y^2}\left[\frac{\log\left(\frac{y}{x}\right)}{2}+\frac{1}{4}\right]=\log|y|+C$$

$$\Rightarrow \frac{x^2}{4y^2}\left[2\log\left(\frac{y}{x}\right)+1\right]+\log|y|=-C$$

$$\Rightarrow x^2\left[2\log\left(\frac{y}{x}\right)+1\right]+4y^2\log|y|=4y^2k$$

[where, $k = -C$] **(1)**

which is the required solution.

38. Solve the following differential equation

$(x^2+1)\frac{dy}{dx}+2xy=\sqrt{x^2+4}$. **All India 2010, 2008**

Given differential equation is

$$(x^2+1)\frac{dy}{dx}+2xy=\sqrt{x^2+4}$$

On dividing both sides by (x^2+1), we get

$$\frac{dy}{dx}+\frac{2xy}{x^2+1}=\frac{\sqrt{x^2+4}}{x^2+1} \quad \text{...(i)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$P=\frac{2x}{x^2+1} \text{ and } Q=\frac{\sqrt{x^2+4}}{x^2+1}$$

$$\therefore \quad \text{IF}=e^{\int \frac{2x}{x^2+1}dx}=e^{\log|x^2+1|}$$

$$\Rightarrow \quad \text{IF}=x^2+1 \quad [\because e^{\log x}=x] \quad \textbf{(1)}$$

$$\left[\because \int \frac{2x}{x^2+1}dx \Rightarrow \text{put } x^2+1=t \Rightarrow 2x\,dx=dt \quad \therefore \int \frac{dt}{t}=\log|t|=\log|x^2+1|\right]$$

Now, solution of this equation is given by

$$y\times \text{IF}=\int (Q\times \text{IF})\,dx+C \quad \textbf{(1)}$$

$$\therefore \quad y(x^2+1)=\int (x^2+1)\cdot\frac{\sqrt{x^2+4}}{x^2+1}dx \quad \textbf{(1)}$$

$$\Rightarrow \qquad y(x^2+1) = \int \sqrt{x^2+4}\,dx$$

$$\Rightarrow \qquad y\,(x^2+1) = \int \sqrt{x^2+(2)^2}\,dx$$

Now, we know that

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$$

$$\therefore \; y(x^2+1) = \frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\log|x+\sqrt{x^2+4}| + C$$

$$\Rightarrow y(x^2+1) = \frac{x}{2}\sqrt{x^2+4} + 2\log|x+\sqrt{x^2+4}| + C$$

which is the required solution. **(1)**

39. Solve the following differential equation

$$(x^3+x^2+x+1)\frac{dy}{dx} = 2x^2+x.$$

HOTS; All India 2010

Firstly, divide given equation by $x^3 + x^2 + x + 1$, then it becomes a variable separable type differential equation and then solve it.

Given differential equation is

$$(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x^2 + x}{x^3 + x^2 + x + 1}$$

It is a variable separable type differential equation.

$$dy = \frac{2x^2 + x}{x^3 + x^2 + x + 1}dx$$

On integrating both sides, we get

$$\int dy = \int \frac{2x^2 + x}{x^3 + x^2 + x + 1}dx$$

$$\Rightarrow \quad y = \int \frac{2x^2 + x}{x^2(x + 1) + 1(x + 1)}dx$$

$$= \int \frac{2x^2 + x}{(x + 1)(x^2 + 1)}dx$$

$$y = I \quad \ldots\text{(i)} \ \mathbf{(1)}$$

where, $$I = \int \frac{2x^2 + x}{(x + 1)(x^2 + 1)}dx$$

Using partial fractions, we get

$$\frac{2x^2 + x}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \quad \ldots\text{(ii)}$$

$$\Rightarrow \quad \frac{2x^2 + x}{(x + 1)(x^2 + 1)} = \frac{A(x^2 + 1) + (Bx + C)(x + 1)}{(x + 1)(x^2 + 1)}$$

$$\Rightarrow \quad 2x^2 + x = A(x^2 + 1) + (Bx + C)(x + 1)$$

Now, comparing coefficients of x^2, x and constant term from both sides, we get constant term from both sides, we get

$$A + B = 2 \quad \text{...(iii)}$$

$$B + C = 1 \quad \text{...(iv)}$$

and $$A + C = 0 \quad \text{...(v)}$$

On subtracting Eq. (iv) from Eq. (iii), we get

$$A - C = 1 \quad \text{...(vi)}$$

On adding Eqs. (v) and (vi), we get

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

On putting $A = \frac{1}{2}$ in Eq. (iii), we get

$$\frac{1}{2} + B = 2 \Rightarrow B = 2 - \frac{1}{2} = \frac{3}{2}$$

On putting $B = \frac{3}{2}$ in Eq. (iv), we get

$$\frac{3}{2} + C = 1 \Rightarrow C = 1 - \frac{3}{2}$$

$$\Rightarrow C = \frac{-1}{2} \quad \textbf{(1)}$$

On substituting the values of A, B and C in Eq. (ii), we get

$$\frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{1/2}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1}$$

On integrating both sides, we get

$$I = \int \frac{2x^2 + x}{(x+1)(x^2+1)} dx = \frac{1}{2}\int \frac{dx}{x+1} + \frac{3}{2}\int \frac{x}{x^2+1} dx - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$\Rightarrow I = \frac{1}{2}\log|x+1| + \frac{3}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1} x + C \quad \textbf{(1)}$$

$$\left[\because \int \frac{x}{x^2+1} dx \Rightarrow \text{put } x^2 + 1 = t \Rightarrow 2x\, dx = dt\right.$$

$$\Rightarrow x\,dx = \frac{dt}{2}, \text{ then } \int \frac{x}{x^2+1}dx =$$

$$\left.\frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log t = \frac{1}{2}\log|x^2+1| + C\right]$$

On putting above value of I in Eq. (i), we get

$$y = \frac{1}{2}\log|x+1| + \frac{3}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1}x + C$$

which is the required solution. **(1)**

40. Solve the following differential equation $\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$. All India 2010

Given differential equation is

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \sqrt{(1+x^2)+y^2(1+x^2)} = -xy\frac{dy}{dx}$$

$$\Rightarrow \quad \sqrt{(1+x^2)(1+y^2)} = -xy\frac{dy}{dx}$$

$$\Rightarrow \quad \sqrt{1+x^2}\cdot\sqrt{1+y^2} = -xy\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{y}{\sqrt{1+y^2}}dy = -\frac{\sqrt{1+x^2}}{x}dx \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{\sqrt{1+x^2}}{x^2}\cdot x\,dx$$

On putting $1+y^2 = t$ and $1+x^2 = u^2$

$$\Rightarrow \quad 2y\,dy = dt \text{ and } 2x\,dx = 2u\,du$$

$$\Rightarrow \quad y\,dy = \frac{dt}{2} \text{ and } x\,dx = u\,du \quad \textbf{(1)}$$

$$\therefore \quad \frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\int \frac{u}{u^2-1}\cdot u\,du$$

$$\Rightarrow \quad \frac{1}{2}\int t^{-1/2}dt = -\int \frac{u^2}{u^2-1}du$$

$$2 \int \quad\quad \int u^2 - 1$$

$\Rightarrow \quad \frac{1}{2}\frac{t^{1/2}}{1/2} = -\int \frac{(u^2-1+1)}{u^2-1}du$ **(1)**

$\Rightarrow \quad t^{1/2} = -\int \frac{u^2-1}{u^2-1}du - \int \frac{1}{u^2-1}du$

$\Rightarrow \quad \sqrt{1+y^2} = -\int du - \int \frac{1}{u^2-(1)^2}du$

$[\because 1+y^2 = t]$

$\Rightarrow \quad \sqrt{1+y^2} = -u - \frac{1}{2}\log\left|\frac{u-1}{u+1}\right| + C$

$\left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$

$\Rightarrow \sqrt{1+y^2} = -\sqrt{1+x^2} - \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C$

which is the required solution. **(1)**

41. Find the particular solution of the differential equation satisfying the given condition $x^2dy + (xy + y^2)\,dx = 0$, when $y(1) = 1$.

Delhi 2010

Given differential equation is

$$x^2dy + (xy + y^2)\,dx = 0$$

Since, degree of each term is same, so the above equation is a homogeneous equation. This equation can be written as

$$x^2dy = -(xy + y^2)\,dx$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-(xy+y^2)}{x^2}$ ...(i)

On putting $y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$ **(1)**

in Eq (i), we get

$$v + x\frac{dv}{dx} = \frac{-(vx^2 + v^2x^2)}{x^2} = -(v + v^2)$$

$$\Rightarrow \quad x\frac{dv}{dx} = -v - v^2 - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -v^2 - 2v$$

$$\Rightarrow \quad \frac{dv}{v^2 + 2v} = \frac{-dx}{x} \qquad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{dv}{v^2 + 2v} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dv}{v^2 + 2v + 1 - 1} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dv}{(v+1)^2 - (1)^2} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{v+1-1}{v+1+1}\right| = -\log|x| + C$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{v}{v+2}\right| = -\log|x| + C$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{\frac{y}{x}}{\frac{y}{x}+2}\right| = -\log|x| + C, \quad \left[\begin{array}{l}\because y = vx \\ \therefore v = \frac{y}{x}\end{array}\right]$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{y}{y+2x}\right| = -\log|x| + C \qquad \text{...(ii)}$$

Also, given that $y = 0$ at $x = 1, y = 1$.

On putting $x = y = 1$ in Eq. (ii), we get

$$\therefore \quad \frac{1}{2}\log\left|\frac{1}{1+2}\right| = -\log 1 + C$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{1}{3}\right| = -\log 1 + C$$

$$\Rightarrow \quad C = \frac{1}{2}\log\frac{1}{3} \quad [\because \log 1 = 0] \quad \textbf{(1)}$$

On putting the value of C in Eq. (ii), we get

$$\frac{1}{2}\log\left|\frac{y}{y+2x}\right| = -\log|x| + \frac{1}{2}\log\frac{1}{3}$$

$$\Rightarrow \quad \log\left|\frac{y}{y+2x}\right| = -2\log|x| + \log\frac{1}{3}$$

$$\Rightarrow \quad \log\frac{y}{y+2x} = \log x^{-2} + \log\frac{1}{3}$$

$$[\because n\log m = \log m^n]$$

$$\Rightarrow \quad \log\frac{y}{y+2x} = \log\frac{1}{x^2} + \log\frac{1}{3}$$

$$\Rightarrow \quad \log\left(\frac{y}{y+2x}\right) = \log\frac{1}{3x^2}$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad \frac{y}{y+2x} = \frac{1}{3x^2}$$

$$\Rightarrow \quad y \cdot 3x^2 = y + 2x$$

$$\Rightarrow \quad y(1-3x^2) = -2x$$

$$\therefore \quad y = \frac{2x}{3x^2-1}$$

which is the required particular solution. **(1)**

42. Find the particular solution of the differential equation satisfying the given condition

$\frac{dy}{dx} = y\tan x$, given that $y = 1$, when $x = 0$.

Delhi 2010

Given differential equation is

$$\frac{dy}{dx} = y \tan x$$

It can be written as $\frac{dy}{y} = \tan x \, dx$ **(1)**

On integrating both sides, we get

$$\int \frac{dy}{y} = \int \tan x \, dx$$

$\Rightarrow \quad \log|y| = \log|\sec x| + C \quad$ ...(i) **(1)**

$$\left[\because \int \frac{1}{y} dy = \log|y| \text{ and } \int \tan x \, dx = \log|\sec x|\right]$$

Also, given that $y = 1$, when $x = 0$.

On putting $x = 0$ and $y = 1$ in Eq.(i), we get

$$\log 1 = \log(\sec 0°) + C$$

$\Rightarrow \quad 0 = \log 1 + C \quad [\because \sec 0° = 1]$ **(1)**

$\Rightarrow \quad C = 0 \quad [\because \log 1 = 0]$

On putting $C = 0$ in Eq. (i), we get the required particular solution as

$$\log|y| = \log|\sec x|$$

$\therefore \quad y = \sec x$ **(1)**

which is the required solution.

43. Solve the following differential equation

$\cos^2 x \frac{dy}{dx} + y = \tan x.$

All India 2009; Delhi 2008, 2011, 2008C

Given differential equation is

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

On dividing both sides by $\cos^2 x$, we get

$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$$

dy ... 2 ... 2 ...

$$\Rightarrow \quad \frac{dy}{dx} + y \cdot \sec^2 x = \tan x \cdot \sec^2 x \quad \ldots(i)$$

$$\left[\because \frac{1}{\cos^2 x} = \sec^2 x\right]$$

which is the linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \ldots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \sec^2 x \text{ and } Q = \tan x \cdot \sec^2 x \quad \textbf{(1)}$$

$$\therefore \quad \text{IF} = e^{\int \sec^2 x \, dx} = e^{\tan x}$$

$$\left[\because \int \sec^2 x \, dx = \tan x + C\right] \textbf{(1)}$$

Now, solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) \, dx + C$$

$$\therefore \quad y \times e^{\tan x} = \int \tan x \cdot \sec^2 x \cdot e^{\tan x} \, dx \ldots(iii)$$

On putting $\tan x = t$

$\Rightarrow \quad \sec^2 x \, dx = dt$ in Eq. (iii), we get

$$\therefore \quad ye^{\tan x} = \int \underset{\text{I}}{t} \, \underset{\text{II}}{e^t} \, dt \quad \textbf{(1)}$$

$$\Rightarrow \quad ye^{\tan x} = t \int e^t dt - \int \left[\frac{d}{dt}(t) \int e^t dt\right] dt$$

[using integration by parts in $\int te^t dt$]

$$\Rightarrow \quad ye^{\tan x} = te^t - \int 1 \times e^t dt$$

$$\Rightarrow \quad ye^{\tan x} = te^t - e^t + C$$

$$\therefore \quad ye^{\tan x} = \tan x \cdot e^{\tan x} - e^{\tan x} + C \; [\because \tan x = t]$$

On dividing both sides by $e^{\tan x}$, we get

$$y = \tan x - 1 + Ce^{-\tan x}$$

which is the required solution. **(1)**

44. Solve the following differential equation

$\sec x \frac{dy}{dx} - y = \sin x.$ **All India 2009C**

Given differential equation is

$$\sec x \frac{dy}{dx} - y = \sin x$$

On dividing both sides by sec x, we get

$$\frac{dy}{dx} - \frac{y}{\sec x} = \frac{\sin x}{\sec x}$$

$$\Rightarrow \quad \frac{dy}{dx} - y\cos x = \sin x \cos x \quad \ldots(i)$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \ldots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = -\cos x \text{ and } Q = \sin x \cos x \quad \textbf{(1)}$$

$$\therefore \quad \text{IF} = e^{\int -\cos x\, dx} = e^{-\sin x}$$

$$[\because \int \cos x\, dx = \sin x + C] \quad \textbf{(1)}$$

Now, solution of above equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C$$

$$\therefore \quad ye^{-\sin x} = \int \sin x \cos x\, e^{-\sin x}\, dx$$

On putting $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore \quad ye^{-\sin x} = \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^{-t}}\, dt \quad \textbf{(1)}$$

$$\Rightarrow \quad ye^{-\sin x} = t\int e^{-t}dt - \int\left[\frac{d}{dt}(t)\int e^{-t}dt\right]dt$$

[using integration by parts]

$$\Rightarrow \quad ye^{-\sin x} = -te^{-t} - \int 1 \times (-e^{-t})dt$$

$$= -te^{-t} + \int e^{-t}dt$$

$$\Rightarrow \quad ye^{-\sin x} = -te^{-t} - e^{-t} + C$$

$$\Rightarrow \quad ye^{-\sin x} = -\sin x\, e^{-\sin x} - e^{-\sin x} + C$$

$$[\because \sin x = t]$$

$$y = -\sin x - 1 + C\, e^{\sin x} \quad \textbf{(1)}$$

which is the required solution.

45. Solve the following differential equation

$$(x\log x)\frac{dy}{dx}+y=2\log x.$$ **Delhi 2009, 2009C**

Given differential equation is

$$(x\log x)\frac{dy}{dx}+y=2\log x$$

On dividing both sides by $x\log x$, we get

$$\frac{dy}{dx}+\frac{y}{x\log x}=\frac{2}{x} \quad \text{...(i)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$P=\frac{1}{x\log x} \text{ and } Q=\frac{2}{x} \quad \textbf{(1)}$$

$$\therefore \quad \text{IF}=e^{\int\frac{1}{x\log x}dx}=e^{\log\log x}$$

$$=\log x \quad [\because e^{\log x}=x] \quad \textbf{(1)}$$

$$\left[\because \int\frac{1}{x\log x}dx \Rightarrow \text{put } \log x=t \Rightarrow \frac{1}{x}dx=dt\right.$$

$$\left.\therefore \int\frac{1}{x\log x}dx=\int\frac{dt}{t}=\log|t|=\log|\log x|\right]$$

Now, solution of above equation is given by

$$y\times\text{IF}=\int(Q\times\text{IF})\,dx+C$$

$$\therefore \quad y\times\log x=\int\underset{\text{II}}{\frac{2}{x}}\underset{\text{I}}{\log x}\,dx \quad \textbf{(1)}$$

$$\Rightarrow \quad y\log x=\log x\int\frac{2}{x}dx$$

$$-\int\left[\frac{d}{dx}(\log x)\int\frac{2}{x}dx\right]dx$$

[using integration by parts]

$$\Rightarrow \quad y \log x = \log x \cdot 2 \log x - \int \frac{1}{x} \cdot 2 \log x \, dx$$

$$\left[\because \int \frac{1}{x} dx = \log|x| + C\right]$$

$$\Rightarrow \quad y \log x = 2 (\log x)^2 - 2 \int \frac{\log x}{x} dx$$

$$\Rightarrow \quad y \log x = 2 (\log x)^2 - \frac{2(\log x)^2}{2} + C$$

$$\left[\text{in} \int \frac{\log x}{x} dx, \text{ put } \log x = t \Rightarrow \frac{1}{x} dx = dt \right.$$

$$\left. \therefore \int t \, dt = \frac{t^2}{2} = \frac{(\log x)^2}{2} + C\right]$$

$$y = 2 (\log x) - (\log x) + \frac{C}{\log x}$$

[dividing both sides by log x] **(1)**

which is the required solution.

46. Solve the following differential equation

$$x \frac{dy}{dx} = y - x \tan\left(\frac{y}{x}\right).$$ All India 2009

Given differential equation is

$$x \frac{dy}{dx} = y - x \tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x \tan\left(\frac{y}{x}\right)}{x} \quad \ldots(i)$$

which is a homogeneous differential equation.

On putting $y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$ **(1)**

in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{vx - x\tan v}{x} = v - \tan v$$

$\Rightarrow \quad x\frac{dv}{dx} = -\tan v$

$\Rightarrow \quad \frac{dv}{\tan v} = -\frac{dx}{x}$ **(1)**

$\Rightarrow \quad \cot v\, dv = -\frac{dx}{x} \quad \left[\because \frac{1}{\tan v} = \cot v\right]$ **(1)**

On integrating both sides, we get

$$\int \cot v\, dv = -\int \frac{dx}{x}$$

$\Rightarrow \quad \log|\sin v| = -\log|x| + C$

$[\because \int \cot v\, dv = \log|\sin v| + C]$

$\Rightarrow \quad \log|\sin v| + \log|x| = C$

$\Rightarrow \quad \log|x\sin v| = C$

$[\because \log m + \log n = \log mn]$

$\therefore \quad \log\left|x\sin\frac{y}{x}\right| = C \quad \left[\because v = \frac{y}{x}\right]$ **(1)**

which is the required solution.

47. Solve the following differential equation

$(1 + x^2)\frac{dy}{dx} + y = \tan^{-1} x.$ **Delhi 2009**

The given differential equation is

$$(1+x^2)\frac{dy}{dx}+y=\tan^{-1}x$$

On dividing both sides by $(1+x^2)$, we get

$$\frac{dy}{dx}+\frac{y}{1+x^2}=\frac{\tan^{-1}x}{1+x^2} \qquad \text{...(i)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q \qquad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$P=\frac{1}{1+x^2} \text{ and } Q=\frac{\tan^{-1}x}{1+x^2} \qquad \textbf{(1)}$$

$$\therefore \qquad \text{IF}=e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$$

$$\left[\because \int\frac{1}{1+x^2}dx=\tan^{-1}x+C\right] \qquad \textbf{(1)}$$

Now, solution of above equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C$$

$$\therefore \quad y \times e^{\tan^{-1} x} = \int \frac{\tan^{-1} x}{1+x^2} \cdot e^{\tan^{-1} x} dx \qquad \text{...(iii)}$$

On putting $\tan^{-1} x = t$

$$\Rightarrow \quad \frac{1}{1+x^2} dx = dt \qquad \textbf{(1)}$$

in Eq. (iii), we get

$$ye^{\tan^{-1} x} = \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^t}\, dt$$

$$\Rightarrow \quad ye^{\tan^{-1} x} = t \int e^t dt - \int \left[\frac{d}{dt}(t) \int e^t dt \right] dt$$

[using integration by parts]

$$\Rightarrow \quad ye^{\tan^{-1} x} = te^t - \int 1 \times e^t dt$$

$$\Rightarrow \quad ye^{\tan^{-1} x} = te^t - e^t + C$$

$$\Rightarrow \quad ye^{\tan^{-1} x} = \tan^{-1} x \cdot e^{\tan^{-1} x} - e^{\tan^{-1} x} + C$$

On dividing both sides by $e^{\tan^{-1} x}$, we get

$$y = \tan^{-1} x - 1 + Ce^{-\tan^{-1} x} \qquad \textbf{(1)}$$

which is the required solution.

48. Solve the following differential equation

$\frac{dy}{dx} + y = \cos x - \sin x.$ **Delhi 2009**

Given differential equation is

$$\frac{dy}{dx} + y = \cos x - \sin x \quad ...(i)$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad ...(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = 1 \text{ and } Q = \cos x - \sin x \quad \textbf{(1)}$$

$$\therefore \quad \text{IF} = e^{\int 1 dx} = e^x \quad \textbf{(1)}$$

Now, solution of above equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C$$

$$\therefore \quad ye^x = \int e^x(\cos x - \sin x)\, dx$$

$$\Rightarrow \quad ye^x = \int \underset{II}{e^x} \underset{I}{\cos x}\, dx - \int e^x \sin x\, dx$$

$$\Rightarrow \quad ye^x = \left[\cos x \int e^x dx - \int \left\{\frac{d}{dx}(\cos x) \int e^x dx\right\} dx\right] - \int e^x \sin x\, dx$$

[applying integration by parts in the first integral]

$$\Rightarrow \quad ye^x = [e^x \cos x - \int - \sin x \cdot e^x dx] - \int e^x \sin x\, dx \quad \textbf{(1)}$$

$$\Rightarrow \quad ye^x = e^x \cos x + \int e^x \sin x\, dx - \int e^x \sin x\, dx + C$$

$$\Rightarrow \quad ye^x = e^x \cos x + C$$

On dividing both sides by e^x, we get

$$y = \cos x + Ce^{-x}$$

which is the required solution. **(1)**

49. Solve the following differential equation
$\frac{dy}{dx} + 2y \tan x = \sin x.$ **All India 2008**

Given differential equation is

$$\frac{dy}{dx} + 2y \tan x = \sin x \quad \text{...(i)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$P = 2 \tan x \text{ and } Q = \sin x \quad \textbf{(1)}$$

$$\therefore \quad \text{IF} = e^{\int 2 \tan x \, dx} = e^{2 \log|\sec x|}$$

$$= e^{\log \sec^2 x} = \sec^2 x \quad \textbf{(1)}$$

Now, solution of above equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) \, dx + C$$

$$y \sec^2 x = \int \sin x \cdot \sec^2 x \, dx$$

$$\Rightarrow \quad y \sec^2 x = \int \frac{\sin x}{\cos^2 x} dx \quad \textbf{(1)}$$

$$\Rightarrow \quad y \sec^2 x = \int \sec x \tan x \, dx$$

$$\left[\because \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x\right]$$

$$\Rightarrow \quad y \sec^2 x = \sec x + C$$

$$\left[\because \int \sec x \tan x \, dx = \sec x + C\right]$$

$$\therefore \quad y = \frac{1}{\sec x} + \frac{C}{\sec^2 x}$$

$$\Rightarrow \quad y = \cos x + C \cos^2 x \quad \textbf{(1)}$$

which is the required solution.

50. Solve the following differential equation
$x^2 \frac{dy}{dx} = y^2 + 2xy.$ **All India 2008**

Given differential equation is

$$x^2 \frac{dy}{dx} = y^2 + 2xy$$

which is a homogeneous differential equation as degree of each term is same in the equation.

Above equation can be written as

$$\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2} \quad \ldots(i)$$

On putting $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \textbf{(1)}$$

in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2 + 2vx^2}{x^2} = v^2 + 2v$$

$$\Rightarrow v + x\frac{dv}{dx} = v^2 + 2v$$

$$\Rightarrow x\frac{dv}{dx} = v^2 + 2v - v \Rightarrow x\frac{dv}{dx} = v^2 + v$$

$$\Rightarrow \frac{dv}{v^2 + v} = \frac{dx}{x} \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int \frac{dv}{v^2 + v} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{v^2 + v + \frac{1}{4} - \frac{1}{4}} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{\left(v + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2 \times \frac{1}{2}} \log \left| \frac{v + \frac{1}{2} - \frac{1}{2}}{v + \frac{1}{2} + \frac{1}{2}} \right| = \log|x| + C$$

$$\left[\because \int \frac{dx}{\quad} = \frac{1}{\quad} \log\left|\frac{x - a}{\quad}\right|\right] \quad \textbf{(1)}$$

$\left[\quad x^2 - a^2 \quad 2a \quad |x + a| \right]$

$$\Rightarrow \quad \log\left|\frac{v}{v+1}\right| - \log|x| = C$$

$$\Rightarrow \quad \log\left|\frac{v}{(v+1)\cdot x}\right| = C$$

$$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \quad \log\left|\frac{\frac{y}{x}}{\left(\frac{y}{x}+1\right)x}\right| = C \qquad \left[\begin{array}{l}\because y = vx \\ \therefore v = \frac{y}{x}\end{array}\right]$$

$$\Rightarrow \quad \log\left|\frac{y}{xy + x^2}\right| = C \qquad \textbf{(1)}$$

which is the required solution.

51. Solve the following differential equation $(x^2 - y^2)\,dx + 2xy\,dy = 0$, given that $y = 1$, when $x = 1$. **Delhi 2008**

Given differential equation is

$(x^2 - y^2)\,dx + 2xy\,dy = 0$

which is a homogeneous differential equation as degree of each term is same.

Above equation can be written as

$$(x^2 - y^2)\,dx = -2xy\,dy \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \quad \dots\text{(i)}$$

On putting $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ **(1)**

in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2vx^2} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$$

$$= \frac{v^2 - 1 - 2v^2}{2v} = \frac{-1 - v^2}{2v} \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{2v}{v^2 + 1} = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{2v}{v^2 + 1} dv = -\int \frac{dx}{x}$$

On putting $v^2 + 1 = t \quad \Rightarrow 2v\,dv = dt$

$$\therefore \quad \int \frac{dt}{t} = -\log|x| + C$$

$$\Rightarrow \quad \log|t| = -\log|x| + C$$

$$\Rightarrow \quad \log|v^2 + 1| + \log|x| = C \quad [\because t = v^2 + 1]$$

$$\Rightarrow \quad \log\left|\frac{y^2}{x^2} + 1\right| + \log|x| = C \quad \text{...(ii)}$$

$$\left[\because v = \frac{y}{x}\right] \textbf{(1)}$$

Also, given that $y = 1$, when $x = 1$.

On putting $x = 1$ and $y = 1$ in Eq. (ii), we get

$$\log 2 + \log 1 = C \quad \Rightarrow \quad C = \log 2 \quad [\because \log 1 = 0]$$

On putting $C = \log 2$ in Eq. (ii), we get

$$\log\left|\frac{y^2 + x^2}{x^2}\right| + \log x = \log 2$$

$$\Rightarrow \quad \log\left|x\left(\frac{x^2 + y^2}{x^2}\right)\right| = \log 2$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad \log\left|\frac{x^2 + y^2}{x}\right| = \log 2 \Rightarrow x^2 + y^2 = 2x \quad \textbf{(1)}$$

which is the required solution.

52. Solve the following differential equation

$$\frac{dy}{dx} = \frac{x(2y - x)}{x(2y + x)}, \text{ if } y = 1, \text{ when } x = 1.$$

Delhi 2008

Given differential equation is

$$\frac{dy}{dx} = \frac{x(2y - x)}{x(2y + x)} \Rightarrow \frac{dy}{dx} = \frac{2xy - x^2}{2xy + x^2} \quad \ldots(i)$$

which is a homogeneous differential equation because each term of numerator and denominator have same degree.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ **(1)**

in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{2vx^2 - x^2}{2vx^2 + x^2} = \frac{2v-1}{2v+1}$$

$$v + x\frac{dv}{dx} = \frac{2v-1}{2v+1}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v-1}{2v+1} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v - 1 - 2v^2 - v}{2v+1}$$

$$\frac{2v+1}{2v^2 - v + 1}dv = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{2v+1}{2v^2 - v + 1}dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad I = -\log|x| + C \quad \ldots(ii)$$

where, $I = \int \frac{2v+1}{2v^2 - v + 1}dv$

Let $2v + 1 = A \cdot \frac{d}{dv}(2v^2 - v + 1) + B$

$$\Rightarrow \quad 2v + 1 = A(4v - 1) + B \quad \ldots(iii)$$

On comparing coefficients of v and constants from both sides, we get

$$4A = 2$$

$$\Rightarrow \qquad A = \frac{1}{2} \text{ and } -A + B = 1$$

$$\Rightarrow \qquad -\frac{1}{2} + B = 1 \Rightarrow B = \frac{3}{2}$$

On putting $A = \frac{1}{2}$ and $B = \frac{3}{2}$ in Eq. (iii), we get

$$2v + 1 = \frac{1}{2}(4v - 1) + \frac{3}{2} \qquad \textbf{(1)}$$

On integrating both sides, we get

$$I = \int \frac{2v + 1}{2v^2 - v + 1} dv$$

$$\Rightarrow I = \int \frac{\frac{1}{2}(4v - 1) + \frac{3}{2}}{2v^2 - v + 1} dv$$

$$\Rightarrow I = \frac{1}{2} \int \frac{4v - 1}{2v^2 - v + 1} dv + \frac{3}{2} \int \frac{dv}{2v^2 - v + 1}$$

$$\Rightarrow I = \frac{1}{2} \log|2v^2 - v + 1| + \frac{3}{4} \int \frac{dv}{v^2 - \frac{v}{2} + \frac{1}{2}}$$

$$\left[\begin{array}{l} \because \int \frac{4v - 1}{2v^2 - v + 1} dv \Rightarrow \text{put } 2v^2 - v + 1 = t \\ \qquad\qquad\qquad\qquad \Rightarrow (4v - 1)\, dv = dt \\ \text{then} \int \frac{dt}{t} = \log|t| = \log|2v^2 - v + 1| \end{array}\right]$$

$$\Rightarrow I = \frac{1}{2} \log|2v^2 - v + 1| + \frac{3}{4} \int \frac{dv}{v^2 - \frac{1}{2}v + \frac{1}{2} + \frac{1}{16} - \frac{1}{16}}$$

$$= \frac{1}{2} \log|2v^2 - v + 1| + \frac{3}{4} \int \frac{dv}{\left(v - \frac{1}{4}\right)^2 + \frac{7}{16}}$$

$$= \frac{1}{2}\log|2v^2 - v + 1| + \frac{3}{4}\int \frac{dv}{\left(v - \frac{1}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2}$$

(1/2)

$$\Rightarrow I = \frac{1}{2}\log|2v^2 - v + 1| + \frac{3}{4} \times \frac{4}{\sqrt{7}} \tan^{-1}\left(\frac{v - \frac{1}{4}}{\frac{\sqrt{7}}{4}}\right)$$

$$\left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$$

$$\Rightarrow I = \frac{1}{2}\log|2v^2 - v + 1| + \frac{3}{\sqrt{7}}\tan^{-1}\left(\frac{4v - 1}{\sqrt{7}}\right)$$

On putting the value of I in Eq. (ii), we get

$$\frac{1}{2}\log|2v^2 - v + 1| + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{4v - 1}{\sqrt{7}}\right) = -\log|x| + C$$ **(1/2)**

$$\Rightarrow \frac{1}{2}\log\left|\frac{2y^2}{x^2} - \frac{y}{x} + 1\right| + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{\frac{4y}{x} - 1}{\sqrt{7}}\right) = -\log|x| + C \quad \left[\because \text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow \frac{1}{2}\log\left|\frac{2y^2}{x^2} - \frac{y}{x} + 1\right| + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{4y - x}{\sqrt{7} \cdot x}\right) = -\log|x| + C \quad \text{...(iv)}$$

Also, given that $y = 1$, when $x = 1$.

On putting $x = 1$ and $y = 1$ in Eq. (iv), we get

$$\frac{1}{2}\log|2| + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{3}{\sqrt{7}}\right) = -\log 1 + C$$

$$\Rightarrow \frac{1}{2}\log 2 + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{3}{\sqrt{7}}\right) = C \quad [\because \log 1 = 0]$$

$\Rightarrow \frac{1}{2}\log 2 + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{3}{\sqrt{7}}\right) = C$

On putting the value of C in Eq. (iv), we get

$$\frac{1}{2}\log\left(\frac{2y^2 - xy + x^2}{x^2}\right) + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{4y - x}{\sqrt{7}x}\right)$$

$$= -\log|x| + \frac{1}{2}\log 2 + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{3}{\sqrt{7}}\right)$$

$$\Rightarrow \log\left(\frac{2y^2 - xy + x^2}{x^2}\right)^{1/2} + \log x - \log(2)^{1/2}$$

$$= \frac{3\sqrt{7}}{7}\left[\tan^{-1}\left\{\frac{\frac{3}{\sqrt{7}} - \left(\frac{4y - x}{\sqrt{7}x}\right)}{1 + \frac{3}{\sqrt{7}} \cdot \left(\frac{4y - x}{\sqrt{7}x}\right)}\right\}\right]$$

$$\left[\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A - B}{1 + AB}\right)\right] \quad \textbf{(1/2)}$$

$$\Rightarrow \log(2y^2 - xy + x^2)^{1/2} - \log\sqrt{2}$$

$$= \frac{3\sqrt{7}}{7}\tan^{-1}\left[\frac{(4x - 4y)\cdot\sqrt{7}}{4x + 12y}\right]$$

$$\left[\because \log\left(\frac{2y^2 - xy + x^2}{x^2}\right) = \log(2y^2 - xy + x^2)^{1/2} = \log(2y^2 - xy + x^2)^{1/2} - \log x\right]$$

$$\Rightarrow \log\sqrt{\frac{2y^2 - xy + x^2}{2}}$$

$$= \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{(x - y)\cdot\sqrt{7}}{x + 3y}\right)$$

$$\Rightarrow \quad \log\sqrt{\frac{2y^2 - xy + x^2}{2}}$$

$$= \frac{3\sqrt{7}}{7}\tan^{-1}\left[\frac{\sqrt{7}x - \sqrt{7}y}{x + 3y}\right] \qquad \textbf{(1/2)}$$

which is the required solution.

6 Marks Questions

53. Find the particular solution of the differential equation $(3xy + y^2)dx + (x^2 + xy)dy = 0$, for $x = 1$ and $y = 1$. **Delhi 2013C**

Given differential equation is

$$(3x^2 + y^2)dx + (x^2 + xy)dy = 0$$

It can be rewritten as $\frac{dy}{dx} = -\frac{3xy + y^2}{x^2 + xy}$ ...(i)

which is a homogeneous differential equation of degree 2.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

in Eq.(i), we get $v + x\frac{dv}{dx} = -\frac{3vx^2 + v^2x^2}{x^2 + vx^2}$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{3v + v^2}{1 + v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\left(\frac{3v + v^2 + v + v^2}{1 + v}\right) \qquad \textbf{(1)}$$

$$\Rightarrow x\frac{dv}{dx} = -\left(\frac{2v^2 + 4v}{1 + v}\right) \Rightarrow \frac{(1 + v)dv}{2(v^2 + 2v)} = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{1 + v}{2(v^2 + 2v)}dv = -\int \frac{dx}{x} \qquad \text{...(ii)} \ \textbf{(1)}$$

Again, put $v^2 + 2v = z \Rightarrow (2v + 2)dv = dz$

$$\Rightarrow \quad (1 + v)dv = \frac{dz}{2}$$

Then, Eq. (ii) becomes,

$$\int \frac{1}{2} \times \frac{dz}{2z} = -\int \frac{dx}{x} \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{1}{4} \log|z| = -\log|x| + \log|C|$$

$$\Rightarrow \quad \frac{1}{4} [\log|z| + 4\log|x|] = \log|C|$$

$$\Rightarrow \quad \log|zx^4| = 4\log|C|$$

$$\Rightarrow \quad zx^4 = C^4 = C_1 \, zx^4 = C_1$$

where, $C_1 = C^4$

$$\Rightarrow \quad x^4(v^2 + 2v) = C_1 \quad [\text{put } z = v^2 + 2v]$$

$$\Rightarrow \quad x^4\left(\frac{y^2}{x^2} + \frac{2y}{x}\right) = C_1 \left[\text{put } v = \frac{y}{x}\right] \ldots \text{(iii)} \quad \textbf{(1)}$$

Also, given that $y = 1$ for $x = 1$.

On putting $x = 1$ and $y = 1$ in Eq. (iii), we get

$$1\left(\frac{1}{1} + \frac{2}{1}\right) = C_1$$

$$\Rightarrow \quad C_1 = 3 \quad \textbf{(1)}$$

Also, given that $y = 1$ for $x = 1$.

So, on putting $C_1 = 3$ in Eq. (iii), we get

$$x^4\left(\frac{y^2}{x^2} + \frac{2y}{x}\right) = 3 \quad \Rightarrow \quad y^2x^2 + 2yx^3 = 3 \quad \textbf{(1)}$$

which is the required particular solution.

54. Show that the differential equation $2ye^{x/y}dx + (y - 2xe^{x/y})dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $x=0$, when $y=1$.

HOTS; Delhi 2013

Firstly, replace x by λx and y by λy in $f(x, y)$ of given differential equation to check that it is homogeneous. If it is homogeneous, then put $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ and then solve.

Given differential equation is $2y\,e^{x/y}dx + (y - 2x\,e^{x/y})dy = 0$. It can be written as

$$\frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}} \quad \ldots(i)$$

$$\text{Let } F(x, y) = \frac{\left(2xe^{\frac{x}{y}} - y\right)}{\left(2ye^{\frac{x}{y}}\right)}$$

On replacing x by λx and y by λy both sides, we get

$$F(\lambda x, \lambda y) = \frac{\left(2\lambda x\, e^{\frac{\lambda x}{\lambda y}} - \lambda y\right)}{\left(2\lambda y\, e^{\frac{\lambda x}{\lambda y}}\right)}$$

$$\Rightarrow \quad F(\lambda x, \lambda y) = \frac{\lambda(2xe^{x/y} - y)}{\lambda(2ye^{x/y})} = \lambda^0\,[F(x, y)] \quad \textbf{(1)}$$

Thus, $F(x, y)$ is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation. **(1)**

To solve it, put $x = vy$

$$\Rightarrow \quad \frac{dx}{dy} = v + y\frac{dv}{dy} \quad \textbf{(1/2)}$$

in Eq.(i), we get $v+y\frac{dv}{dy}=\frac{2ve^{v}-1}{2e^{v}}$

$$\Rightarrow y\frac{dv}{dy}=\frac{2ve^{v}-1}{2e^{v}}-v=\frac{2ve^{v}-1-2ve^{v}}{2e^{v}}$$

$$\Rightarrow \quad 2e^{v}dv=\frac{-dy}{y} \quad \textbf{(1)}$$

On integrating both sides, we get

$$\int 2e^{v}\,dv=-\int\frac{dy}{y} \Rightarrow 2e^{v}=-\log|y|+C$$

Now, replace v by $\frac{x}{y}$, we get

$$2e^{x/y}+\log|y|=C \quad \text{...(ii)} \; \textbf{(1½)}$$

Also, given that $x=0$, when $y=1$.

On substituting $x=0$ and $y=1$ in Eq. (ii), we get

$$2e^{0}+\log|1|=C \Rightarrow C=2$$

On substituting the value of C in Eq. (ii), we get

$$2e^{x/y}+\log|y|=2$$

which is the required particular solution of the given differential equation. **(1)**

55. Show that the differential equation $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\left(\frac{y}{x}\right)=0$ is homogeneous.

Find the particular solution of this differential equation, given that $x=1$, when $y=\frac{\pi}{2}$. **Delhi 2013**

Given differential equation is

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)=y\sin\left(\frac{y}{x}\right)-x \Rightarrow \frac{dy}{dx}=\frac{y}{x}-\frac{1}{\sin\frac{y}{x}}$$

...(i)

$\left[\text{dividing both sides by } x\sin\left(\frac{y}{x}\right)\right]$

Let $$(x, y)=\frac{y}{x}-\frac{1}{\sin\frac{y}{x}}$$

On replacing x by λx and y by λy on both sides, we get

$$F(\lambda x, \lambda y)=\frac{\lambda y}{\lambda x}-\frac{1}{\sin\frac{\lambda y}{\lambda x}} =\lambda^0\left(\frac{y}{x}-\frac{1}{\sin\frac{y}{x}}\right)$$

$$=\lambda^0 F(x, y)$$

So, given differential equation is homogeneous.

(2)

On putting $y=vx$

$\Rightarrow$ $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in Eq.(i), we get **(1)**

$$v+x\frac{dv}{dx}=v-\frac{1}{\sin v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1}{\sin v} \Rightarrow \quad \sin v\, dv = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int \sin v\, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \qquad -\cos v = -\log|x| + C$$

$$\Rightarrow -\cos y/x = -\log|x| + C \quad \left[\because v = \frac{y}{x}\right]$$ **(1½)** ...(ii)

Also, given that $x = 1$, when $y = \frac{\pi}{2}$.

On putting $x = 1$ and $y = \frac{\pi}{2}$ in Eq. (ii), we get

$$-\cos\left(\frac{\pi}{2}\right) = -\log|1| + C$$

$$\Rightarrow \qquad -0 = -0 + C \Rightarrow C = 0$$

On putting the value of C in Eq. (ii), we get

$$\cos\frac{y}{x} = \ln|x|$$

which is the required solution. **(1½)**

56. Find the particular solution of the differential equation $\frac{dx}{dy} + x\cot y = 2y + y^2 \cot y$, $(y \neq 0)$, given that $x = 0$, when $y = \frac{\pi}{2}$.

All India 2013

Given differential equation is

$$\frac{dx}{dy} + x\cot y = 2y + y^2 \cot y, (y \neq 0)$$

which is a linear differential equation.

On comparing with $\frac{dx}{dy} + Px = Q$, we get

$$P = \cot y \text{ and } Q = 2y + y^2 \cot y$$

$\therefore$ IF $= e^{\int P dy} = e^{\int \cot y\, dy} = e^{\log \sin y} = \sin y$ **(1½)**

Now, the solution of above differential equation is given by

$$x \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dy + C$$

$$\therefore \quad x \sin y = \int (2y + y^2 \cot y) \sin y\, dy + C$$

$$= 2\int y \sin y\, dy + \int \underset{\text{I}}{y^2} \underset{\text{II}}{\cos y}\, dy + C$$

$$= 2\int y \sin y\, dy + y^2 \int \cos y\, dy$$

$$- \int \left[\left(\frac{d}{dy} y^2 \right) \int \cos y\, dy \right] dy + C$$

[using integration by parts in second integral]

$$= 2\int y \sin y\, dy + y^2 \sin y - 2\int y \sin y\, dy + C$$

$$= y^2 \sin y + C$$

$\Rightarrow \quad x \sin y = y^2 \sin y + C$ ...(i) **(2)**

Also, given that $x = 0$, when $y = \frac{\pi}{2}$.

On putting $x=0$ and $y=\frac{\pi}{2}$ in Eq. (i), we get

$$0=\left(\frac{\pi}{2}\right)^2 \sin\frac{\pi}{2}+C \Rightarrow C=-\frac{\pi^2}{4}$$ **(1/2)**

On putting the value of C in Eq. (i), we get

$$x\sin y=y^2\sin y-\frac{\pi^2}{4} \Rightarrow x=y^2-\frac{\pi^2}{4}\cdot \operatorname{cosec} y$$

which is required particular solution of given differential equation. **(2)**

57. Show that the differential equation $\left[x\sin^2\left(\frac{y}{x}\right)-y\right]dx+xdy=0$ is homogeneous.

Find the particular solution of this differential equation, given that $y=\frac{\pi}{4}$, when $x=1$. All India 2013

Given differential equation is

$$\left[x\sin^2\left(\frac{y}{x}\right)-y\right]dx+xdy=0$$

$$\Rightarrow \qquad \frac{dy}{dx}=\frac{y-x\sin^2\left(\frac{y}{x}\right)}{x} \qquad \ldots(i)$$

Let $\qquad F(x,y)=\dfrac{y-x\sin^2\left(\frac{y}{x}\right)}{x}$

On replacing x by λx and y by λy both sides, we get

$$F(\lambda x,\lambda y)=\frac{\lambda\left[y-x\sin^2\left(\frac{y}{x}\right)\right]}{\lambda\, x}=\lambda^0\,[F(x,y)]$$

Thus, given differential equation is a homogeneous differential equation. **(1)**

On putting $y=vx \Rightarrow \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$ in Eq. (i), we get

$$v+x\frac{dv}{dx}=\frac{vx-x\sin^2\left(\frac{vx}{x}\right)}{x}$$

$$\Rightarrow \quad v+x\frac{dv}{dx}=v-\sin^2 v \quad \Rightarrow \quad x\frac{dv}{dx}=-\sin^2 v$$

$$\Rightarrow \qquad \text{cosec}^2 v\,dv=\frac{-dx}{x} \qquad \textbf{(2)}$$

On intergrating both sides, we get

$$\int \text{cosec}^2 v\,dv+\int\frac{dx}{x}=0$$

$$\Rightarrow \qquad -\cot v+\log|x|=C$$

$$\Rightarrow \qquad -\cot\left(\frac{y}{x}\right)+\log|x|=C \quad \left[\because v=\frac{y}{x}\right]\ldots(ii)$$

Also, given that, $y=\dfrac{\pi}{4}$, when $x=1$.

On putting $x = 1$ and $y = \frac{\pi}{4}$, in Eq. (ii), we get

$$-\cot\left(\frac{\pi}{4}\right) + \log|1| = C \quad \textbf{(2)}$$

$$\Rightarrow \quad C = -1 \quad \left[\because \cot\frac{\pi}{4} = 1\right]$$

On putting the value of C in Eq. (ii), we get

$$-\cot\left(\frac{y}{x}\right) + \log|x| = -1$$

$$\Rightarrow \quad 1 + \log|x| - \cot\left(\frac{y}{x}\right) = 0$$

which is the required particular solution of given differential equation. **(1)**

58. Find the particular solution of the differential equation $(\tan^{-1} y - x)dy = (1 + y^2)dx$, given that $x = 0$, when $y = 0$. All India 2013

Given differential equation is

$$(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$$

$$\Rightarrow \quad \frac{\tan^{-1} y - x}{1 + y^2} = \frac{dx}{dy} \Rightarrow \frac{dx}{dy} = \frac{-x}{1 + y^2} + \frac{\tan^{-1} y}{1 + y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1 + y^2} \cdot x = \frac{\tan^{-1} y}{1 + y^2}$$

which is a linear differential equation of first order. **(1)**

On comparing with $\frac{dx}{dy} + Px = Q$, we get

$$P = \frac{1}{1 + y^2} \quad \text{and} \quad Q = \frac{\tan^{-1} y}{1 + y^2}$$

$\therefore \quad IF = e^{\int P dy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$ **(1)**

Now, solution of above differential equation is given by

$$x \cdot (IF) = \int Q \cdot (IF)\, dy + C$$

$\Rightarrow \quad xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \times e^{\tan^{-1} y} + C$ **(1)**

On putting $t = \tan^{-1} y \Rightarrow dt = \frac{1}{1+y^2} dy$

$\therefore \quad x \cdot e^{\tan^{-1} y} = \int t \cdot e^t \, dt + C$

$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = t \cdot e^t - \int 1 \cdot e^t dt + C$

[using integration by parts]

$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = t \cdot e^t - e^t + C$

$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = (\tan^{-1} y - 1)\, e^{\tan^{-1} y} + C$ ...(i) **(1)**

Also, given that, when $x = 0$, then $y = 0$.

On putting $x = 0, y = 0$ in Eq. (i), we get

$$0 = (\tan^{-1} 0 - 1) e^{\tan^{-1} 0} + C$$

$\Rightarrow \quad 0 = (0-1)\, e^0 + C \Rightarrow 0 = (0-1) \cdot 1 + C$

$\Rightarrow \quad C = 1$ **(1)**

On putting the value of C in Eq. (i), we get

$\therefore \quad x \cdot e^{\tan^{-1} y} = (\tan^{-1} y - 1) \cdot e^{\tan^{-1} y} + 1$

$\Rightarrow \quad x = \tan^{-1} y - 1 + e^{-\tan^{-1} y}$

which is the required particular solution of the differential equation. **(1)**